Critical points and "phase transitions" in the stochastic behavior of a nonautonomous anharmonic oscillator

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Transitions of new types occur in the stochastic behavior of simple dynamic systems in the particular case of an anharmonic oscillator whose potential energy is a periodic function of the time. In particular, a chaos-chaos transition is observed.

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1. The anharmonic oscillator, a simple model in nonlinear physics, has reattracted the strong interest of theoreticians and experimentalists,[1–4] primarily because of the observations of a nontrivial stochastic behavior in physical systems which are simple in the sense that they have only a few degrees of freedom. The stochastic dynamics results from a pronounced instability of the individual finite motions in the nonlinear system and is totally unrelated to the penetration into the system of anything "truly random"—noise or fluctuations. Thanks to the active research by physicists and mathematicians, it is now fair to say that the primary paths have been identified for the transitions of a determinate system from regular oscillations to a stochastic regime.[5,6] On the other hand, very little is known about the changes in the properties of the stochastic regime itself upon changes in the parameters. For example, the onset of a stochastic regime (the growth of fluctuations, the broadening of the spectrum, etc.) in systems with many degrees of freedom is usually associated with simply an increase in the number of modes involved in the nonlinear interaction as the deviation of the system from equilibrium increases.

In this letter we show that qualitative changes can occur in the properties of the stochastic motion even in a very simple system and that these changes are accompanied at the critical points by important changes in the way in which the characteristics of the motion depend on a parameter. Among these characteristics of the stochastic motion are (a) the parameter which reflects the extent of the coupling of normal variables in the regime of steady-state stochastic oscillations (a quantitative parameter might be the fractal dimensionality of the stochastic set in phase space)[8] and (b) the intensity of the stochastic motion (e.g., the integrated power spectrum of the oscillations). Furthermore, it may prove useful to also describe the attractors in terms of the degree to which the stochastic set in phase space is localized near regular trajectories of a certain type: periodic trajectories, separatrices, etc.[9]

2. We have studied the case of a parametrically excited anharmonic oscillator,

$$u_{tt} + ku_t + (1 + q\cos\Omega t)u + u^3 = 0, \qquad (1)$$

which exhibits[3] a stochastic behavior over broad ranges of the parameters k and q. We have (numerically) analyzed the Poincaré mapping of the secant plane $t = \text{const}$ onto

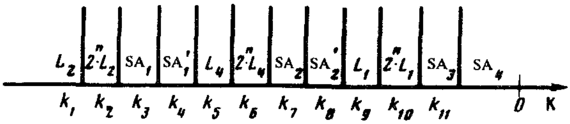

FIG. 1. Sequence of transitions through the critical points k_j upon a decrease in the dissipation. L_i—stable periodic motion with a period iT; $2^n L_i$—infinite chain of bifurcations involving a doubling of the periodic motion L_i; SA—strange attractor.

itself with a period $T = 2\pi/\Omega$. While varying k and q, we monitored the power spectrum $P(\omega)$, the average power $N = \int P(\omega)d\omega$, and the fractal dimensionality $D = 2 + |\lambda^+/(k + \lambda^+)|$, where λ^+ is the positive Lyapunov characteristic index averaged over the stochastic set (the Kolmogorov-Sinai entropy[7]).

Figure 1 shows the primary transitions observed in stochastic motion (1) upon a decrease in the parameter k. We observed transitions of three main types: 1) the emergence from a region where trajectories run close together of an unstable and a stable periodic motion corresponding to a regime of regular oscillations ($k = k_4$, $k = k_8$); 2) the appearance of a chain of doublings of the period of the stable periodic motion ($k_1 \leqslant k < k_2$, $k_5 \leqslant k < k_6$, $k_9 \leqslant k < k_{10}$) which gives rise to a strange attractor ($k = k_2$, $k = k_6$, $k = k_{10}$); 3) an abrupt increase in the size of the region with the strange attractor ($k = k_3$, $k = k_7$, $k = k_{11}$), accompanied by an abrupt change in the integrated power spectrum. Here the transitions through the critical points k_3, k_7, and k_{11} are nontrivial; we will restrict the present analysis to the transition through k_{11}.

3. To reach an understanding of the mechanism for this transition we need to know the "past history" of the behavior of the oscillator as the dissipation decreased from k_3 to k_{10}. By the "time" at which k becomes equal to k_{11}, a well-developed stochastic set B already exists in the phase space of the nonautonomous oscillator, but this set is not attractive. The "interrelations" between this nonattractive set and the strange attractor A_3 (Fig. 1) at $k \approx k_{11}$ determine the phenomena in which we are interested here.

At $k = k_8$, two periodic motions are created from a region where the trajectories run close together: a saddle motion Γ_1 and a stable motion L_1; the latter loses its stability at $k = k_9$, passing on its stability to a periodic motion with twice the period. In the interval $k_9 < k < k_{10}$ there is an infinite chain of period-doubling bifurcations, which culminates in the creation of a strange attractor[1)] A_3. The attraction region of this attractor is bounded by the stable separatrix Γ_1. With a further decrease in the dissipation, $k \lesssim k_{11}$, the attractor A_3 swells, absorbing in succession the unstable periodic motions[2)] Γ^s which arose from the period-doubling bifurcations as k decreased from k_9 to k_{10}. At $k \lesssim k_{11}$, a homoclinic structure B_3 appears on the basis of Γ_1; this homoclinic structure combines with the structures B_1 and B_2, which arose earlier (Fig. 1), forming a nonattractive stochastic set B. This set (in contrast with attractor A_3) covers a broad region in the phase space; furthermore, it corresponds to some extremely fast motions (in the structures $B_{1,2}$ and B_3). At $k \gtrsim k_{11}$, the A_3 boundary intersects the stable separatrix Γ_1, and in this manner attractor A_3 combines with B, forming a symmetric positioned attractor A_4 (Fig. 2).[3)]

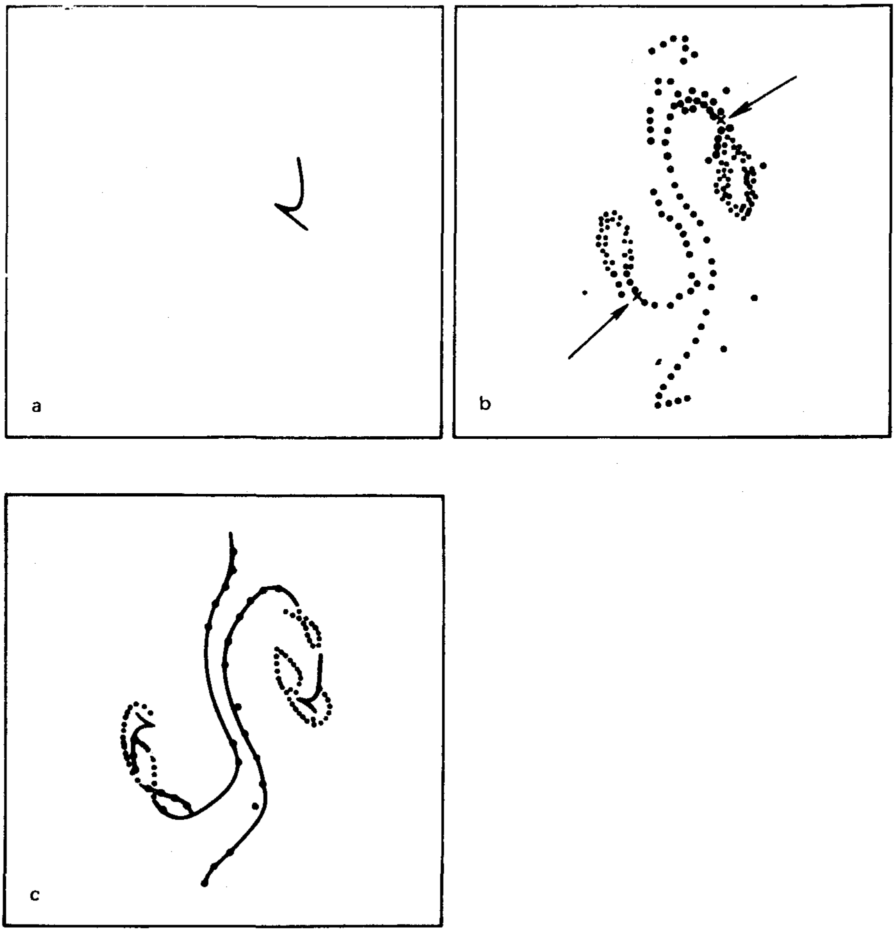

FIG. 2. a—Portrait of the strange attractor on the secant u, u_t after a period $T = 2\pi/\Omega$ (the vertical dimensions are $|u_t| < 50$; (the horizontal dimensions are $|u| \leqslant 10$; the number of periods is $t/T \sim 3 \times 10^3$; and $k = 0.4576$); b—nonattractive stochastic set B ($k = 0.4576$; the arrows indicate periodic saddle motions Γ_1 and $\overline{\Gamma}_1$); c—strange attractor after the "phase transition" ($k = 0.4575$).

It is the absorption of the broad stochastic set B by the highly localized attractor A_3 which causes the "jump" in the average fluctuation power and the fractal dimensionality of the strange attractor (Fig. 3).

The motion of the oscillator immediately after the appearance of A_4 is very peculiar: The image point spends most of its time in the region occupied previously by A_3, only rarely entering B, where it spends only a short time because of the pronounced instability of the trajectories. An oscillogram of the motion thus takes the form of low-intensity stochastic oscillations interrupted by highly "turbulent" spikes: a chaos-chaos intermittence.

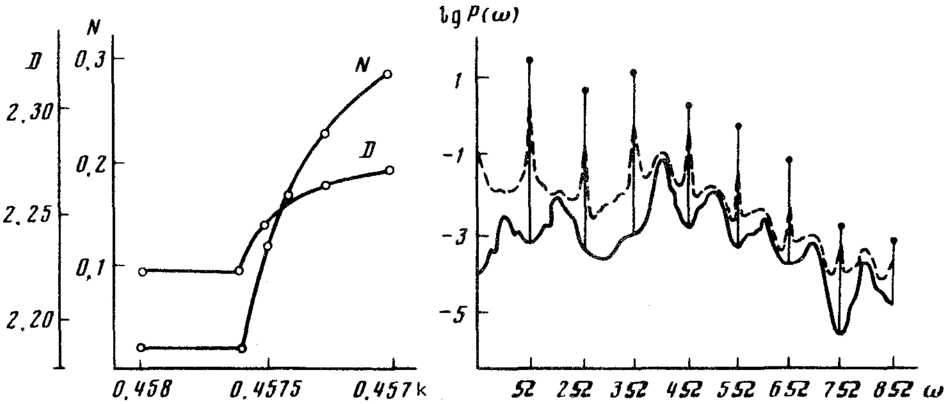

FIG. 3. a—Change in the average power and in the fractal dimensionality of the strang attractor near the transition ($q = 50$, $\Omega = 2.04$); b—power spectrum of the variable before the transition ($k = 0.4576$) and immediately after it ($k = 0.4575$, or $\Delta k = 10^{-4}$; the dashed curve).

We wish to emphasize that these transitions of the type k_{11} are accompanied by substantial changes in all the basic characteristics of the stochastic motion. We have already mentioned the jumps in the dimensionality of the strange attractor and the average power; there is also a change in the nature of the oscillations: Before the transition, the fluctuations are associated with a random walk of the image point exclusively in the localization region of A_3, while after the transition the attractor lies near a new structure, which is also determined by B.

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1) The motion on this attractor corresponds to asymmetric oscillations of the oscillator—on secant $u, u_t > 0$. The symmetry of (1) can be seen in the fact that while A_3 exists there also exists a mirror-image attractor $\overline{A}_3$, which is localized in the region $u, u_t < 0$. Whether one of these attractors is actually realized depends on the initial conditions.

2) The boundary of attractor A_3 can be taken approximately to be the unstable separatrix of that periodic saddle motion from Γ^s which has the shortest period and which has not yet been absorbed by the attractor.

3) The possibility in principle of the absorption by an attractor of an unstable stochastic set in a different situation (for an attractor of the Lorentz type) was pointed out in Ref. 10.

[1] P. S. Linsay, Phys. Rev. Lett. **47**, 1349 (1981).
[2] J. S. Testa, J. Perez, and C. Jeffries, Phys. Rev. Lett. **48**, 714 (1981).
[3] F. M. Izrailev, M. I. Rabinovich, and A. D. Ugodnikov, Phys. Lett. **87A**, 16 (1981).
[4] P. Holmes, Philos. Trans. R. Soc. London **A292**, 1394 (1979).
[5] M. I. Rabinovich, Usp. Fiz. Nauk **125**, 123 (1978) [Sov. Phys. Usp. **21**, 443 (1978)].
[6] P. Eckmann, Rev. Mod. Phys. **53**, 643 (1981).
[7] I. P. Kornfel'd, Ya. G. Sinaĭ, and S. V. Fomin, Érgodicheskaya teoriya (Ergodic Theory), Nauka, Moscow, 1980.
[8] F. Ledrappier, Commun. Math. Phys. **81**, 229 (1981).
[9] M. A. Gol'shtik and V. N. Shtern, Dokl. Akad. Nauk SSSR **237**, 1319 (1981) [Sov. Phys. Dokl. **26**, 557 (1981)].
[10] V. S. Afraĭmovich, V. V. Bykov, and P. P. Shil'nikov, Trudy MMO **44**, 150 (1982).

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