On the stochastic properties of relativistic cosmological models near the singularity

E. M. Lifshitz, I. M. Khalatnikov, Ya. G. Sinai, K. M. Khanin and L. N. Shchur
Institute for Physical Problems, L. D. Landau Institute for Theoretical Physics, Academy of Sciences USSR

(Submitted 10 June 1983)
Pis'ma Zh. Eksp. Teor. Fiz. **38**, No. 2, 79–82 (25 July 1983)

It is shown that the quantitative parameters of the previously developed[1] statistical theory of oscillatory evolution of cosmological models in the proximity to singularity can be calculated in exact manner.

PACS numbers: 98.80.Dr

The oscillatory mode of approach towards the singularity was first discovered for the homogeneous vacuum cosmological model of Bianchi type IX (see Ref. 2). The character of the evolution of a model can be described by indicating three "scale functions" $a(t)$, $b(t)$, $c(t)$ which determine the temporal evolution of the lengths in three different directions in space. The oscillatory mode consists of an infinite sequence of successive periods (in Ref. 2 they were called eras) during which two of the scale functions oscillate and the third one decreases monotonically. On passing from one era to another (with decreasing time t) the monotonic decrease in transferred to another of the three scale functions. The amplitude of oscillations increases during each era but the increase is especially strong on passing from one era to another; however, the product abc decreases montonically—approximately as t. The eras become condensed with $t \to 0$; an adequate temporal variable for description of their replacements appears to be the "logarithmic time" $\Omega = -\ln t$.

We denote by k_0, k_1, k_2,... the "lengths" of successive eras (measured in terms of the number of oscillations they contain), beginning from a certain initial one. It turns out that this sequence of the lengths is determined by a sequence of the number x_{-1}, x_0, x_1, x_2,... $(0 < x_s < 1)$, each of which arises from the preceding one by the transformation

$$x_{s+1} = \{ 1 / x_s \}, \tag{1}$$

where the curly brackets denote the fractional part of the number. The lengths $k_s = [1/x_{s-1}]$, the square brackets denoting the integer part of the number. It was pointed out by I. M. Lifshitz and the two of us[1] that the law of replacements of the lengths of the eras according to (1) leads to an important property: spontaneous stochastization of the behavior of the model on approach to singularity ($t = 0$) and the "loss of memory" of the initial conditions, prescribed at some instant of time $t = t_0 > 0$ (this paper is cited henceforth as I).

The importance of the oscillatory evolution in the homogeneous models stems from the fact that this model serves as a prototype for construction of a general inhomogeneous solution of the Einstein equations (in the neighborhood of the singularity); the relevant work has been recently reviewed in Ref. 3. Although the inhomo-

 0021-3640/83/140091-04$01.00

geneity and the presence of matter give rise to the appearance of certain new feaιures (rotation of the axes to which the scale functions a,b,c refer), the law (1) remains unlatered. Thus, stochasticity in the vicinity of the singularity appears to be a most general property of cosmological models based on the classical Einstein equations.

The knowledge of the source of the stochastization makes it possible to construct with a considerable completeness a statistical theory of the evolution of the cosmological model in asymptotic proximity to singularity. However, for a calculation of parameters of this theory an approximate method was devised in I, the degree of exactness of which is difficult to estimate beforehand. The aim of the present work is to show that these parameters can be calculated exactly.

The starting point of the theory is the formula due to Gauss, $w(x) = 1/(1+x)\ln 2$, which determines the probability distribution density of the values of $x_s \equiv x$ in the interval [0,1] after many iterations of the transformation (1) (as we shall speak—in the stationary, i.e., independent of s, limit)[1]. Hence follows the formula

$$W(k) = \frac{1}{\ln 2} \ln \frac{(k+1)^2}{k(k+2)}$$

for the probability distribution of the integer values of the era lengths. This function decreases with $k \to \infty$ merely as k^{-2}; such slowness makes it necessary to use logarithmic physical quantities in order to obtain for them stable statistical distributions and mean values.

The basis of the following analysis constitutes the recurrence formulas (obtained in I) for successive eras:

$$\Omega_{s+1}/\Omega_s = 1 + \delta_s k_s (k_s + x_s + 1/x_s) \equiv \exp \xi_s , \qquad (2)$$

$$\delta_{s+1} = 1 - \frac{(k_s/x_s + 1)\,\delta_s}{1 + \delta_s k_s (k_s + x_s + 1/x_s)} ; \qquad (3)$$

They are valid in asymptotic limit when $\ln\Omega/\Omega \to 0$ [in I formula (5) was given with a misprint in the denominator]. Here Ω_s is the instant of the beginning of the sth era; the quantity δ_s is the measure (in units of Ω_s) of the initial (in the same era) amplitude α_s of the oscillations of the logarithms of the scale functions ($\ln a$, $\ln b$, $\ln c$): $\alpha_s = \delta_s \Omega_s$ ($0 \leqslant \sigma_s \leqslant 1$). The quantity δ_s has a stable stationary statistical distribution $P(\delta)$ and a stable (small relative fluctuations) mean value. For their determination in I was used (with due reservation) an approximate method based on the assumption of statistical independence of the random quantity δ_s of the random quantities k_s, x_s. Now an exact solution of this problem is given.

Since we are interested in statistical properties in the stationary limit, it is reasonable to introduce the so-called natural expansion of the transformation (1) by continuing it without limit to negative indices. Such a "doubly-infinite" sequence $X = (\ldots, x_{-1}, x_0, x_1, x_2, \ldots)$ is uniform in its statistical properties over its entire length (and x_0 loses its meaning of an "initial" condition). The sequence X is equivalent to a sequence of integers $K = (..., k_{-1}, k_0, k_1, k_2...)$, constructed by the rule $k_s = [1/x_{s-1}]$. Inversely, every number of X is determined by the integers of K as an infinite continuous fraction

$$x_s = 1/(k_{s+1} + 1/(k_{s+2} + 1/(k_{s+3} + ...) \equiv x^+_{s+1}.$$

We also introduce the quantities which are defined by a continuous fraction with a retrograde sequence of the denominators

$$x^-_s = 1/(k_{s-1} + 1/(k_{s-2} + 1/(k_{s-3} + ...).$$

By means of some rearrangements (3) can be brought to the form

$$x_s(1-\delta_{s+1})/\delta_{s+1} = 1/(k_s + x_{s-1}(1-\delta_s)/\delta_s).$$

Hence by iterations: $x_s(1-\delta_{s+1})/\delta_{s+1} = x^-_{s+1}$ and finally $\delta_s = x_s^+/(x_s^+ + x_s^-)$.

The quantities x_s^+ and x_s^- have a joint stationary distribution $P(x^+, x^-)$ which can be found starting from the joint transformation

$$x^+_{s+1} = \{1/x^+_s\}, \quad x^-_{s+1} = 1/([1/x^+_s] + x^-_s). \qquad (4)$$

In contrast to (1) it is a one-to-one mapping (in the unit square of variation of x^+ and x^-). Therefore the condition for the distribution to be stationary is expressed simply by the equation

$$P(x^+_{s+1}, x^-_{s+1}) = P(x^+_s, x^-_s)J(x^+_s, x^-_s),$$

where J is the Jacobian of the transformation (4). The normalized solution of this equation is

$$P(x^+, x^-) = 1/(1 + x^+x^-)^2 \ln 2 \qquad (5)$$

[its integration over x^+ or x^- yields $w(x)$]. Since δ_s is expressed in terms of x_s^+ and x_s^-, the knowledge of (5) makes it possible to find the distribution

$$P(\delta) = 1/(|1 - 2\delta| + 1)\ln 2. \qquad (6)$$

The mean value $\langle\delta\rangle = 1/2$ already as a result of the symmetry of this function.

According to I the "doubly-logarithmic" time interval for a succession of a given number s of eras is $\tau_s \equiv \ln(\Omega_s/\Omega_0) = \Sigma\, \xi_p$ (summation from $p = 1$ to $p = s$). The mean value $\langle\tau_s\rangle = s\langle\xi\rangle$. The expression of ξ_s from (2) can be reduced to the form

$$\xi_s = \ln(\delta_s/(1-\delta_{s+1})x_{s-1}x_s).$$

Since $\langle\ln\delta_s\rangle = \langle\ln(1-\delta_{s+1})\rangle$ and $\langle\ln x_{s-1}\rangle = \langle\ln x_s\rangle$, we obtain

$$\langle\xi\rangle = -2\langle\ln x\rangle = \pi^2/6\ln 2 = 2{,}37.$$

For large s the values of τ_s are distributed around $\langle\tau_s\rangle$ according to the Gauss law with the density

$$\rho(\tau_s) = (2\pi D)^{-1/2} \exp\{-(\tau_s - \langle\tau_s\rangle)^2/2D\} \quad (7)$$

(see I§4). The calculation of the variance D is more complicated since it demands not only the knowledge of $\langle\xi^2\rangle$ but also the mean values $\langle\xi_{p_1}\xi_{p_2}\rangle$ (which actually depend only on the difference $p = |p_1 - p_2|$). It appears to be useful to rearrange the terms in the sum $\Sigma\xi_p$ and omit the terms which do not increase with s. Thus one can obtain

$$\sum \xi_p = \sum \ln(1/x_p^+ x_p^-) \equiv \sum \eta_p.$$

The variance

$$D = s\left\{\langle\eta^2\rangle - \langle\eta\rangle^2 + 2\sum_{p=1}^{\infty}(\langle\eta_0\eta_p\rangle - \langle\eta\rangle^2)\right\}.$$

The mean value $\langle\eta\rangle = \langle\xi\rangle$, and for the mean square one can obtain $\langle\eta^2\rangle = 9\zeta(3)/2\ln 2 = 7.80$. Without taking into account correlations we would obtain $D = 2.17s$. By taking into account correlations with $p = 1,2,3,4$ (calculated with an electronic computer) we arrive at the value $D = (3.5 \pm 0.1)s$.

1) The regular evolution of the model according to the rule (1) can be interrupted by the appearance of "anomalous" eras (which were called in Ref. 2 the case of small oscillations). However, it is important that in the asympotic vicinity of the singularity (as $t \to 0$) the probability of occurrence of such "dangerous" cases tends to zero, as was proved in I§4.

2) The reduction of the transformation to the one-to-one mapping was used alredy by Chernoff and Barrow[4]—for other variables and without applications to the problems considered here. As to the preceding papers by Barrow,[5] they contain nothing beyond the main idea (taken from I) about the connection of stochasticity in cosomological models with the transformation (1) and with the distributions $w(x)$ and $W(k)$ (and the repetition of some well-known statements of the general ergodic theory).

[1] E. M. Lifshitz, I. M. Lifshitz, and I. M. Kalatnikov Zh. Eksp. Teor. Fiz. **59**, 322 (1970) [Soviet Physics JETP **32**, 173, 1971].

[2] V. A. Belinskii, I. M. Kalatnikov and E. M. Lifshitz, Adv. Phys. **19**, 525 (1970).

[3] V. A. Belinskii, I. M. Kalatnikov and E. M. Lifshitz, Adv. Phys. **31**, 639 (1982).

[4] D. F. Chernoff and J. D. Barrow, Phys. Rev. Lett. **50**, 134 (1983).

[5] J. D. Barrow, Phys. Rev. Lett. **46**, 963, (1981); Gen. Rel. Grav. **14**, 523 (1982); Phys. Rep. **85C**, 1 (1982).

Translation furnished by the authors