Self-focusing of a magnetosonic wave across the magnetic field

D. Yu. Manin and V. I. Petviashvili
I. V. Kurchatov Institute of Atomic Energy, Moscow

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Simplified equations are derived to describe the evolution of an acoustic-wave beam which is propagating from a steady-state source through a medium with a positive dispersion. It is shown in the particular case of a fast magnetosonic wave that a beam of this type may undergo a self-focusing.

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The nonlinear effects accompanying the propagation of a fast magnetosonic wave were first analyzed by Kaplan and Stanyukovich,[1] who showed that a fast magnetosonic wave breaks in the one-dimensional case, and by Sagdeev,[2] who took the dispersion into account and found a soliton resulting from the breaking. Kadomtsev and Petviashvili[3] later showed that one-dimensional solitons are unstable in a medium with a positive dispersion. The dispersion of fast magnetosonic waves is always positive if the ions have a sufficiently high temperature, satisfying $\beta > m_e/m_i$, where $\beta = 4\pi nT/B^2$. In this case, a three-dimensional packet of fast magnetosonic waves does not form a one-dimensional soliton but instead collapses or spreads out. If, on the other hand, we are dealing with a beam of fast magnetosonic waves which is bounded in the radial direction, then we would naturally expect that at a sufficiently high density the beam would undergo a self-focusing as a light beam does in a nonlinear medium. As we will see below, this effect does in fact occur, but the quantitative description of the self-focusing of fast magnetosonic waves is quite different from that of the self-focusing of light. The difference stems from the different nature of the nonlinearity (the self-focusing of sound in a dispersive medium involves overtones, while that of light involves beats). We will derive equations describing a steady-state beam of fast magnetosonic waves which is propagating across the magnetic field. The dispersion relation for the fast magnetosonic waves under the conditions $B \| z, k_y, k_z \ll k_x$ and $\omega < \omega_B (m_i/m_e)^{1/2}$ is

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$$\omega = k_x c_A [1 + r_B^2 k_x^2 / 2 + (k_y^2 + k_z^2) / 2k_x^2 + r_A^2 k_z^2 / 2] ; \qquad (1)$$

$$c_A^2 = B^2 / 4\pi n m_i ; \quad \omega_B = eB/m_i c ; \quad r_A = c_A / \omega_B ; \quad r_B^2 = T / m \omega_B^2 .$$

We see that at $\omega > \omega_B$ the anisotropy due to the last term comes into play. The main nonlinear corrections to the wave equation for fast magnetosonic waves were found in Refs. 1 and 2. Working from those corrections and from (1), by the methods of Ref. 3. we could derive a simplified equation describing the time evolution of a three-dimensional wave packet, and we could show, by analogy with Ref. 4, that such packets cannot be stable. For the self-focusing problem, however, it is more convenient to alter the derivation of Ref. 3 slightly in order to describe the steady-state propagation of the acoustic beam. In this case we find an equation analogous to the Kadomtsev-Petviashvili equation[3]:

$$(\partial_x h + 2hh' + h''' + \partial_z^2 h' / \beta)' = \Delta_\perp h . \qquad (2)$$

Here we have introduced the dimensionless variables $\tau = x/r_B - \omega_B t$; $x/r_B, y/r_B, z/r_B$ to replace x, y, z; and $h = B_\sim / B$, where $B_\sim$ is the magnetic field of the wave. The prime denotes the derivative with respect to τ. Equation (2) describes the propagation along the x axis of waves from the $x = 0$ boundary, where a periodic boundary condition localized in the y, z plane is specified.

Whitham[5] has shown that a Korteweg-de Vries equation can be simplified for an initial-condition problem by a procedure involving an averaging over the "fast" variable τ. When we perform this averaging in the present case, we must take into account two further conditions: the constancy of the (fundamental) periodic source, which is conserved along the beam, and the vanishing of the average value of h over τ. Accordingly, the average equation is derived here by a slightly different method of successive approximations. Equation (2) has a two-parameter family of one-dimensional periodic solutions, expressed in terms of the Jacobi elliptic functions with square modulus $k^2 = m$ (Ref. 6):

$$h_0 = a \, \mathrm{cn}^2 (b\tau + \theta(x)) + c, \qquad (3)$$

where a, b, $\partial_x \theta$, and c are expressed in terms of m and the fundamental frequency of the source by

$$b = K(m) \omega / \pi ; \quad a = 12 b^2 m ;$$

$$c = - a [E(m) / mK(m) - (1 - m)/m] ; \qquad (4)$$

$$\partial_x \theta = 4b^2 (2m - 1) - c .$$

Here $K(m)$ and $E(m)$ are the complete elliptic integrals, and the frequency ω is expressed in units of $\omega_B \beta^{-1/2}$. As a zeroth approximations of the spatially nonuniform solution (2) we adopt the function h_0, in which we assume m to be dependent on the coordinates x, y, z. In this manner we specify h at the $x = 0$ boundary as the sum of temporal harmonics which contain an elliptic function, so that the self-focusing is monotonic. In this approach the dispersive terms in (2) is on the order of the nonlinear term. Ozhogin *et al.*[7] have analyzed the self-focusing in the case of a slight nonlinearity. In this case, the nonlinearity of course acts more slowly. For the first approximation we find from (2) the following linear equation:

$$h_1 + h_0 h_1 + h_1'' = \Delta_\perp \tilde{\tilde{h}}_0 - \partial_z^2 h_0 - [d\widetilde{h_0}/dx - h_0(b + \partial_x \theta)] \equiv F, \tag{5}$$

where the tilde denotes the inverse transform over τ. This equation has the solution

$$h_1 = h_0' \int (h_0')^{-2} \{\int F h_0' d\tau\} d\tau , \tag{6}$$

which may be bounded only if

$$\langle F h_0' \rangle = 0, \tag{7}$$

where the angle brackets denote an average over τ. This is the average equation which we have been seeking. Substituting F from (5) into (7), and integrating by parts, we find

$$\partial_x < h_0^2 > + \partial_y (< h_0^2 > \partial_y \theta) + \partial_z \{[< h_0^2 > + <(h_0')^2 > / \beta] \partial_z \theta \} = 0. \tag{8}$$

To save space, we will omit the lengthy explicit expressions for the m dependence. We introduce $I(x,y,z) = \langle h_0^2 \rangle = \Phi(m)$, $\partial_x \theta = \Psi(m)$, $\partial_y \theta = v_y(x,y,z)$, $(\langle h_0^2 \rangle + \langle (h_0')^2 \rangle / \beta) \partial_z \theta = v_z(x,y,z)$, $\langle (h_0')^2 \rangle = \Xi(m)$, $v(v_y, v_z)$, $\mathrm{div}_\perp = (\partial_y, \partial_z)$. We then find the following system of equations for I and $\mathbf{v}$:

$$\begin{cases} \partial_x I + \mathrm{div}_\perp (I \mathbf{v}) = 0; \\ \partial_x v_y = \partial_y f(I); \quad \partial_x (v_z g(I)) = \partial_z f(I); \end{cases} \tag{9}$$

where the functions f and g are defined by

$$f(\Phi(m)) = \Psi(m) > 0; \quad g(\Phi(m)) = [1 + \Xi(m) / \beta \Phi(m)]^{-1} . \tag{10}$$

System (9) is reminiscent of the dynamic equation of a compressible gas with a negative pressure. This is a convenient system of equations for numerical calculations; it is more graphic than the original equation and it does not contain the fast variable τ. In particular, at small values of m, $m \ll 1$, we have

$$\begin{aligned} & I = < h_0^2 > \approx \omega^4 m^2 (1 + 4m/3); \quad f(I) \approx 2\sqrt{2I} - 2I/\omega^2 ; \\ & \partial_x \theta \approx \omega^3 (3m - 1), \quad g(I) \approx [1 + (\omega^2 - 2\sqrt{2I}/3) / \beta] . \end{aligned} \tag{11}$$

In this case the system of equations becomes a system of Hamilton's equations. Working from similarity considerations, and using (11), we can easily find estimates of the characteristic length L_y and L_z for the beam contraction along the y and z axes in terms of the dimensionless initial intensity I_0: $L_y \sim R_y (I_0/8)^{1/4}$; $L_z \sim R_z (I_0/8(1 + \omega^2/\beta))^{1/4}$. Figure 1 shows the results of a numerical calculation for a beam of elliptical cross section with a ratio of semiaxes $R_z/R_y = \sqrt{1 + \omega^2/\beta}$, for which $L_y = L_z$, i.e., which is focused to a point. For a plasma in the laboratory the most convenient direction is perpendicular to the magnetic field, for which the dispersion length is at a minimum, so that the source can be small in area. In this case the frequency must be much higher than ω_B. The intensity required for self-focusing turns out to be high, but it can be produced in pulses. As a result of the self-focusing, absorption occurs in a small region near the focus. If the

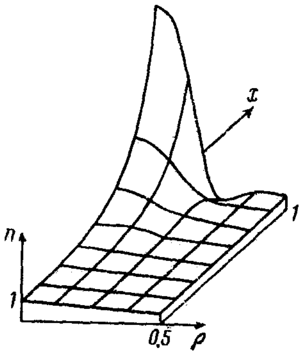

FIG. 1. Central part of a focused beam under the boundary condition $I_0 = \exp(-\rho^2), \rho^2 = y^2 + z^2/(1 + \omega^2/\beta)$ (in dimensionless units).

propagation direction is slightly different from the perpendicular to **B**, energy will be absorbed by waves moving along this slightly nonperpendicular direction. Linear expressions for the absorption are given in Ref. 8, but the nonlinear absorption is apparently more important.

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