Quantization of Hall conductivity

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(Submitted 4 October 1983)
Pis'ma Zh. Eksp. Teor. Fiz. **38**, No. 9, 454–458 (10 November 1983)

A qualitative explanation is given for the experimentally observed quantization of the Hall conductivity σ_{xy} in units $e^2/2\pi\hbar$ for two-dimensional systems in a magnetic field. The explanation is based on the formulation of the localization theory in terms of the renormalization group. The difference between σ_{xy} and $ne^2/2\pi\hbar$ $(\delta\sigma_{xy})$ at finite temperatures is related only to the presence of the dissipative conductivity σ_{xx} $(|\delta\sigma_{xy}| = \alpha\sigma_{xx}$, where α is a number of order 1).

PACS numbers: 73.25. + i

1. The experimental study of the Hall conductivity σ_{xy} in two-dimensional systems in a strong, perpendicular magnetic field H shows that at low temperatures the dependence $\sigma_{xy}(H)$ has a series of plateaus, on which σ_{xy} is an integer multiple of $e^2/2\pi\hbar$. The amazing accuracy with which this quantization is satisfied was already noted in earlier works.[1,2] The efforts of a number of authors[3] achieved a qualitative understanding, but the problem of the theoretical limitations on the accuracy remained unresolved and it was understood that an exhaustive theory of the phenomenon must simultaneously describe both the quantization of σ_{xy} and the Anderson localization of two-dimensional electrons in the random field of impurities.

In the theory of localization, the Q-field method has been developing successfully now for several years. With the help of this method, the calculation of any quantity, for example, the density-density correlation function, reduces to calculating the functional integral[4]

$$K_\omega(\mathbf{r}-\mathbf{r}') = \int DQ(\mathbf{x})\,\frac{1}{4}\,\mathrm{Sp}[(1+\Lambda)Q(\mathbf{r})(1-\Lambda)Q(\mathbf{r}')]\exp\left(\int d\mathbf{x}\,F\right)\Big/\int DQ(\mathbf{x})\exp\left(\int d\mathbf{x}F\right)\Big|_{N=0}, \quad (1)$$

where for an electron in a magnetic field[1] we have

$$\int d\mathbf{r}\,F\{Q\} = \int d\mathbf{r}\ \mathrm{Sp}\left\{-\frac{2\pi\hbar}{e^2}\sigma_{xx}(\vec{\nabla}Q)^2 - \frac{\pi\hbar}{2e^2}\sigma_{xy}\,Q[\nabla_xQ,\nabla_yQ] - i\omega\tau(\Lambda Q)\right\}, \quad (2)$$

where Q is a Hermitian $2N\times 2N$ matrix which satisfies the conditions

$$Q^2 = 1, \quad \mathrm{Sp}\,Q = 0 \quad (3)$$

Λ is a diagonal $2N\times 2N$ matrix, in which the first N numbers along the diagonal are equal to 1 and the remaining numbers are equal to -1.

2. Conditions (3) indicate that all possible values of Q correspond to points of a Grassman manifold $M = U(2N)/[U(N)\times U(N)]$. When the coordinate $\mathbf{r}$ runs through all points in the plane, the values of Q fill some region A in M. The homotopic classification of mappings of the plane onto M is given by the group $\pi_2(M)$. In accordance with the general rules, we have

$$\pi_2(U(2N)/U(N)\times U(N)) = \pi_2(SU(2N)/S[U(N)\times U(N)])$$
$$= \pi_1(S[U(N)\times U(N)]) = \pi_1(SU(N)) + \pi_1(SU(N)) + \pi_1(U(1)/Z_2) = Z. \tag{4}$$

Equation (4) can be interpreted as follows: Q is the generalized density matrix $\rho(\epsilon,\epsilon') = \psi(\epsilon)\psi^*(\epsilon')$, corresponding to two energy levels ϵ and ϵ', $\rho(\epsilon,\epsilon')$ admits a transformation of the phases of the wave functions $\psi(\epsilon)\to\psi(\epsilon)e^{i\phi(\epsilon)}$, $\psi(\epsilon')\to\psi(\epsilon')e^{i\phi(\epsilon')}$, and only the transformations with the differences $\phi(\epsilon)-\phi(\epsilon')$ are physically different. For this reason, the stationary subgroup of the Grassman manifold M always contains the group of changes of phase $U(1)$ and the classification of the mappings of the plane onto M coincides with the classification of mappings of the circle in to $U(1)$. The latter are characterized by a change in φ in circumscribing the circle equal to $2\pi n$, where n is an integer. Thus the matrix function $Q(\mathbf{r})$ gives for all N the mapping of the plane onto M, which is characterized by an integer n (multiplicity of the mapping), which is a topological invariant. Remarkably, for any N[5]

$$\int d\mathbf{r}\,\mathrm{Sp}\,\{Q\,[\nabla_x Q,\ \nabla_y Q]\} = 8\pi\, in. \tag{5}$$

In a field theory with the action (2), all values of σ_{xy}, which differ by an integer multiple of $e^2/2\pi\hbar$, are therefore equivalent.

3. We shall discuss what happens when renormalization is implemented, i.e., when a part Q_0, which is a slowly varying function of the coordinates ($Q = U^+Q_0U$) is singled out in the dependence $Q(\mathbf{r})$, while the integration is performed over the rapidly varying unitary matrices U. In this case, σ_{xx} and σ_{xy} are renormalized, satisfying the equations

$$\frac{d\sigma_{xx}}{d\xi} = \beta_{xx}(\sigma_{xx},\ \sigma_{xy});\quad \frac{d\sigma_{xy}}{d\xi} = \beta_{xy}(\sigma_{xx},\ \sigma_{xy});\quad \xi = \ln\frac{L}{r_H}. \tag{6}$$

Equations (6) describe the dependence of σ_{xx} and σ_{xy} on the linear dimensions of the specimen L at $T=0$. For the initial conditions, determiing σ_{xx} and σ_{xy} for a specimen with dimensions of the order which can be of the magnetic length r_H, we must choose their values which can be calculated with the help of the kinetic equation. Since the action (2) does not change when $e^2/2\pi\hbar$ is added to σ_{xy}, β_{xx} and β_{xy} are periodic functions of σ_{xy} with the period $e^2/2\pi\hbar$. It is also evident that the change in the sign of the magnetic field and, therefore, the change in the sign of σ_{xy} cannot affect the solutions of (6). For this reason, β_{xx} is an even function of σ_{xy} and β_{xy} is an odd function.

Figure 1a illustrates the phase portrait of the system (6). This portrait is periodically repeated in each of the bands $[(e^2/2\pi\hbar)n] < \sigma_{xy} < [(e^2/2\pi\hbar)(n+1)]$. Combining the symmetry relative to the change in sign of σ_{xy} and periodicity with respect to σ_{xy}, we see that the phase portrait in each of the bands is symmetrical relative to the center line $\sigma_{xy} = (e^2/4\pi\hbar)(2n+1)$.

Within the framework of perturbation theory (i.e., when $\sigma_{xx}\gg e^2/\hbar$), σ_{xy} is not renormalized and does not affect the renormalization of σ_{xx}. For this reason, for

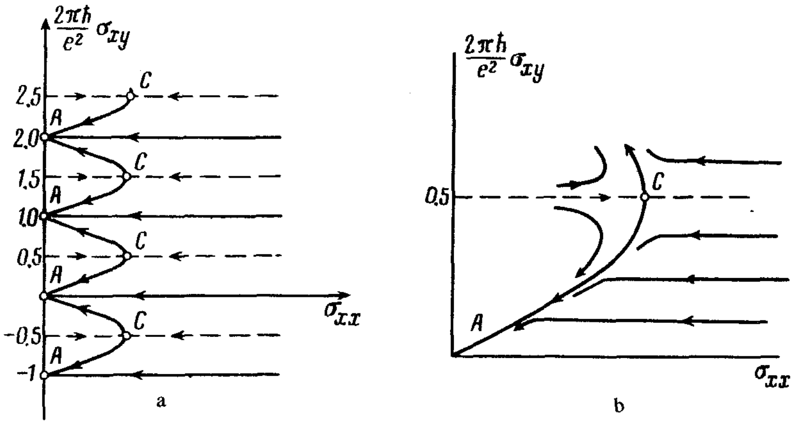

FIG. 1. Separatrices and fixed points (a) and integral curves (b) of the system of renormalization-group equations (6).

$\sigma_{xx} \gg e^2/\hbar$, the integral curves (6) do not depend on σ_{xy} and are illustrated by lines parallel to the abscissa axis. For $\sigma_{xy} = 0$, i.e., in the absence of a magnetic field, localization occurs in a two-dimensional disordered system and, for this reason, the corresponding integral curve (6) coincides with the abscissa axis and terminates at the origin of coordinates, which is the fixed point in Eqs. (6). The points A $(0, e^2 n/2\pi\hbar)$ are also such fixed points. In view of the symmetry, the center line of each band $\sigma_{xy} = (e^2/4\pi\hbar)(2n+1)$ is the integral curve of the system (6). As demonstrated in Ref. 5, nonzero conductivity σ_{xx} corresponds to a half-integer value of $2\pi\hbar\sigma_{xy}/e^2$. In the language of the renormalization group, this means that the straight line $\sigma_{xy} = (e^2/4\pi\hbar)(2n+1)$ has a stable fixed point C with $\sigma_{xx} \neq 0$.

It is natural to assume that the point C is a saddle point and is unstable relative to excursions away from the center line of the band. The separatrix relates C to the stable point A. Then, all integral curves, except the center line of the band, terminate at A, as indicated in Fig. 1b. The results of renormalization of σ_{xy} and σ_{xx} are illustrated schematically in Figs. 2a and 2b. Here, the curves 1 correspond to the dependences of σ_{xy} and σ_{xx} on H, obtained with the help of the kinetic equation. Curves 2 correspond to finite L, while curves 3 correspond to $L \to \infty$.

At finite temperature, $T \neq 0$, inelastic processes, which lead to finite conductivity σ_{xx}, are possible. For this reason, each T corresponds to a length $L(T)$, such that for $L \ll L(T)$, renormalization is implemented according to Eqs. (6), while for $L \gg L(T)$, σ_{xx} and σ_{xy} no longer depend on L. As $T \to 0$, $L(T) \to \infty$. Thus, the curves in Figs. 2a and 2b correspond to the experimental dependences of σ_{xy} and σ_{xx} on H at high temperatures (curves 1), moderately low temperatures (curves 2), and at $T = 0$ (curves 3).

4. To describe the shape of the plateau in the dependences $\sigma_{xy}(H)$, it is necessary to analyze in detail the phase portrait near the point A. This region corresponds to small values of σ_{xx}. We shall therefore examine an electron gas with density $n_e \ll eH/\hbar c$. At $T = 0$, the electrons occupy energy levels corresponding to the lowest part of the Landau level, broadened due to scattering by impurities. The wave func-

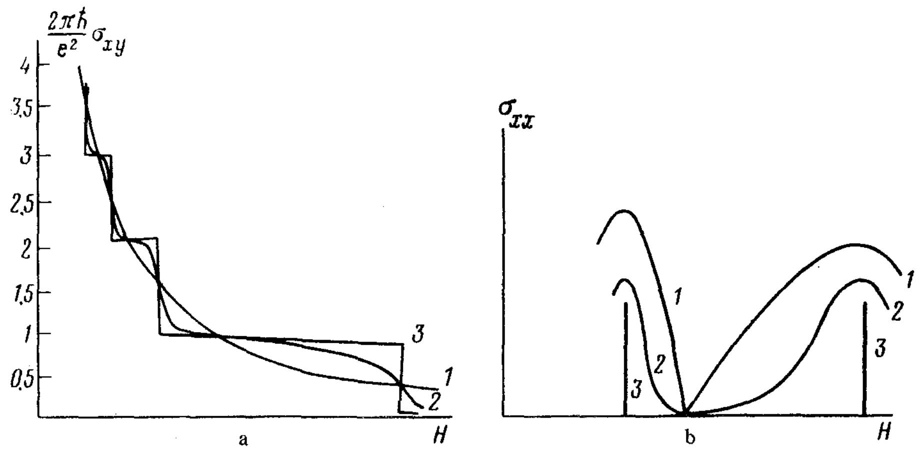

FIG. 2. Results of renormalization of σ_{xx} (a) and σ_{xy} (b).

tions corresponding to these levels are localized. The localization lengths L_c corresponding to them are of the order of r_H and are virtually independent of n_e up to terms of order $\hbar c n_e/eH$. For large linear dimensions of the specimen L, σ_{xx} and σ_{xy} are proportional to e^{-L/L_c}. In this case, the ratio $\alpha = \sigma_{xy}/\sigma_{xx}$ is independent not only of L but also of n_e. This means that the integral curve of the system of equations (6), corresponding to the first Landau subband, is the straight line $\sigma_{xy} = \alpha\sigma_{xx}$, and independent of the electron density, the values of σ_{xx} and σ_{xy} at $L = r_H$ fall on this line. For other subbands with $L = r_H$, the points $(\sigma_{xx}, \sigma_{xy})$ lie to the right of the line $|\sigma_{xy} - e^2n/2\pi\hbar| = \alpha\sigma_{xx}$. In this case, there are two possibilities: either the straight lines $|\sigma_{xy} - e^2n/2\pi\hbar| = \alpha\sigma_{xx}$ near the points A coincide with the stable separatrices and then as $T \to 0$

$$\left| \sigma_{xy} - \frac{e^2 n}{2\pi\hbar} \right| = \alpha\sigma_{yx} \tag{7}$$

as illustrated in Fig. 1b or these separatrices are unstable and then $|\sigma_{xy} - e^2n/2\pi\hbar| : \sigma_{xx}$ depends on the number n.

The dependence (7) for $n = 1$ was observed recently.[6] In this case, in spite of the considerable asymmetry of the plateau, α on the right- and left-sides of its wings coincided and did not depend on T. Variations of α with $n \neq 1$ will make it possible to determine more accurately the form of the separatrix AC, which determines the theoretical accuracy of the measurement of $e^2/\hbar$ with the help of the Hall effect. Numerical simulation could also give important information.

The proposed theory predicts that the relative width of the plateau reaches 100% in the limit $T \to 0$, while the conductivity σ_{xx} with fields H corresponding to a jump in σ_{xy} is equal to the universal value of the order of $e^2/\hbar$.

5. It should be noted that to quantize σ_{xy}, in principle, it is not necessary that the condition $\omega_c \gg 1$ be satisfied. At $\omega_c\tau \ll 1$, it is sufficient that $E_F\omega_c\tau^2/\hbar \gg 1$, but, in this case, $\sigma_{xx} \sim e^2 E_F \tau/\hbar^2 \gg e^2/\hbar$ and, therefore Anderson localization and quantization of σ_{xy} appear only at very low temperatures. Analogously, the use of inversion layers instead of films with a thickness of angstroms is related not to fundamental reasons but only to the possibility of observing the quantization of σ_{xy} at temperatures that are achievable in practice.

The expression for the action (2), strictly speaking, can be derived either with $\omega_c \tau \ll 1$ and $E_F \tau \gg \hbar$, or with $\omega_c \tau \gg 1$, but $E_F/\hbar\omega_c \gg 1$, and assuming that the impurity potential is Gaussian. However, it appears that the result concerning the exact quantization of σ_{xy}, based on topological considerations, is more widely applicable than the rigorous derivation of (2). The use of the electron-electron interaction and of the many-valley carrier spectrum requires a modification of the theory. This will be done in the future.

I am grateful to S. Libby for sending me a preprint and Yu. A. Bychkov, A. I. Larkin, É. I. Rashba, and M. V. Éntin for a discussion of the results.

[1)]The last term on the right side of (2) has been derived in an unpublished paper by Pruisken (cited in Ref. 5).

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Translated by M. E. Alferieff
Edited by S. J. Amoretty