Spontaneous transition to a stochastic state in a four-dimensional Yang-Mills quantum theory

A. M. Semikhatov
P. N. Lebedev Physics Institute, Academy of Sciences of the USSR, Moscow

(Submitted 14 September 1983)
Pis'ma Zh. Eksp. Teor. Fiz. **38**, No. 9, 465–469 (10 November 1983)

The quantum expectation values in a four-dimensional Yang-Mills theory are represented in each topological sector as expectation values over the diffusion which develops in the "fourth" Euclidean time. The Langevin equations of this diffusion are stochastic duality equations in the $A_4 = 0$ gauge.

PACS numbers: 11.15.Tk

1. We wish to outline a new approach to calculating quantum expectation values in a four-dimensional Yang-Mills theory [$YM(R^4)$] in the $A_4 = 0$ gauge.[1)] In this approach, $YM(R^4)$ is represented as the result of a stochastic quantization of some three-dimensional theory whose Lagrangian is a Chern-Simons 3-form (see Appendix 3 in Ref. 1). Specifically, with an external source in $YM(R^3 \times [t', t''])$ we consider a matrix element

$$<A'', t'' | A', t' >^J = <A'' | \exp(-\hat{H}(t''-t')) | A' >^J$$

$$= \int\limits_{A(t')=A',\, A(t'')=A''} DA_i \exp\left\{ -\frac{1}{2} \int\limits_{t'}^{t''} dt \int d^3x \, (\dot{A}_i^a \dot{A}_i^a + B_i^a B_i^a) + \int d^4x \, J_i^a(x) A_i^a(x) \right\}, \tag{1}$$

 0021-3640/83/210565-05$01.00

where $i = 1,2,3$;

$$B_i = \frac{1}{2}\epsilon_{ijk}F_{jk} \equiv \frac{1}{2}\epsilon_{ijk}(\partial_j A_k - \partial_k A_j + g[A_j, A_k]);$$

and $A'(\mathbf{x})$ and $A''(\mathbf{x})$ are given field configurations for which the difference between the Chern-Simons numbers $C' = D[A']$ and $C'' = [A'']$ are determined by the topological charge $q = C'' - C'$ of those fields $A_i(x)$ which contribute in integral (1). Here

$$C[A] = \frac{1}{8\pi^2}\int d^3\mathrm{x}\, \epsilon_{ijk}\,\mathrm{tr}\,(A_i\partial_j A_k + \frac{2}{3}g A_i A_j A_k).$$

We first consider the case $q \geqslant 0$. In the given qth topological sector, the Yang-Mills action is then bounded from below by the value $S_{\mathrm{Cl}} = 8\pi^2 q/g^2$, which is reached only in the case of instanton solution of the duality conditions, which are written in the selected gauge as

$$\dot{A}_i = -B_i.$$

2. The quantum fluctuations around the solutions of these nonlinear classical equations can be described by a transition of these equations to a stochastic nature through the introduction of a random force $\eta_i^a(x, t)$, a white noise, in them. The stochastic duality equations are[2)]

$$\dot{A}_i = \eta_i - B_i, \quad A_i(\mathbf{x}, t') = A_i'(\mathbf{x}). \tag{2}$$

The expectation values for all the η-dependent expressions are calculated in accordance with

$$<\eta^{\otimes(2k+1)}> = 0, \quad <\eta_i^a(\mathbf{x}, x^4)\,\eta_j^b(\mathbf{y}, y^4)> = \delta^{ab}\delta_{ij}\,\delta^{(4)}(x-y)$$

and by means of Wick's theorem for monomials of higher even powers.

Equations (2) may be thought of as Langevin equations describing a diffusion with an additive noise and a potential drift force

$$B_i^a[A(\mathbf{x})] = -8\pi^2\,\delta C[A]/\,\delta A_i^a(\mathbf{x}).$$

Our basic assertion is that the Green's functions found from (1) agree within a factor $\exp(-S_{\mathrm{Cl}})$ with the correlation functions for the diffusion process (2). More precisely, we have

$$<A'', t''|A', t'>^J = e^{-8\pi^2 q/g^2} <\delta\,(A(t'') - A'')\exp\int_{t'}^{t''} dt \int d^3\mathrm{x}\, A_i^a J_i^a>, \tag{3}$$

where the expectation value on the right is over the diffusion which we have introduced, and A_i^a is expressed in terms of η by means of (2).

To trace the origins of this agreement it is convenient to adopt the compact notation $\phi^i = A_i^a(\mathbf{x})$, in terms of which an equation like (2) can be written

$$\dot{\phi}^i(t) = \eta^i(t) - b^i(\phi), \quad \phi^i(t') = \phi'^i. \tag{4}$$

The expectation value on the right side of (3) now becomes (within a normalization factor)

$$\zeta(J) \equiv \langle \delta(\phi^i(t'') - \phi''^i) \exp \int_{t'}^{t''} \phi^i(t) J^i(t)\, dt \rangle$$

$$= \int D\eta \exp\left(-\frac{1}{2}\int dt\, \eta^i(t)\, \eta^i(t)\right) \delta(\phi(t'') - \phi'') \exp \int_{t'}^{t''} \phi^i(t) J^i(t)\, dt,$$

where $\phi^i(t)$ is found from (4). Following Ref. 3, we now find

$$\zeta(J) = \int_{\phi(t')=\phi',\ \phi(t'')=\phi''} D\phi \exp\left(-\frac{1}{2}\int_{t'}^{t''} dt\, (\dot\phi^i\dot\phi^i + b^i b^i - \partial_i b^i)\right.$$

$$\left. -(S)\int_{t'}^{t''} b^i\, d\phi^i\right) \exp \int_{t'}^{t''} \phi^i(t) J^i(t)\, dt, \qquad (5)$$

where $(S)\int$ denotes the Stratonovich stochastic integral (see Ref. 4 for a definition of this integral, which is motivated by differential-geometry structures; for a discussion of its properties; for a comparison with the Itô integral in the geometric formulation; and for further references). A fundamental property of the (S) integral is the Newton-Leibnitz formula[3)]

$$(S)\int_{t'}^{t''} \partial_i c\, d\phi^i = c(\phi'') - c(\phi').$$

The use of this formula in (5) for the potential drift force $b^i = \partial_i c$, which also satisfies the relation $\partial_i b^i = 0$ in a Yang-Mills theory, completes the proof of Eq. (3).

3. If $q < 0$, then we should begin with the antiduality equation $\dot{A}_i = B_i$. Correspondingly, the sign of $C[A]$ changes in all the equations, so that the quantity $S_{\mathrm{Cl}} = 8\pi^2(C' - C'')/g^2$ is again greater than zero.

Consequently, depending on the topological sector of the theory, we have the equations

$$A_i + \alpha B_i = \eta_i, \qquad (6)$$

with $\alpha = \pm 1$. The value of the topological charge q determines the method for constructing a perturbation theory in (6). Specifically, we adopt $A_i = \overline{A}_i + Q_i$ where $\overline{A}$ is an (anti-) instanton with a given topological charge, and the fluctuations Q against this background are determined as a power series in η.

The 0-instanton sector can be described by any of the two stochastic duality equations (6). In terms of the Fourier components (in the limits $t' \to -\infty, t'' \to +\infty$),

$$A(\mathbf{k}, \omega) = \frac{1}{(2\pi)^4}\int d^4x\, e^{-i(\mathbf{k}\cdot\mathbf{x} + \omega t)} A(\mathbf{x}, t),$$

Eq. (6) can be rewritten in a form suitable for iterations:

$$A_i^a(k) = S_{ij}(k)\,[\,\eta_j^a(k) - \frac{\alpha}{2}\epsilon_{jkl} f^{abc} g \int d^4p A_k^b(k-p) A_l^c(p)]\,,$$

$$S_{ij}(k) = \frac{\omega}{i} D_{ij}(k) + \frac{i\alpha\,\epsilon_{imj} k_m}{\mathbf{k}^2+\omega^2}\,, \quad D_{ij}(k) = \frac{\delta_{ij} + k_i k_j/\omega^2}{\mathbf{k}^2+\omega^2}\,,$$

where we have used the four-dimensional notation $k = (\mathbf{k},\omega)$. For the binary correlation function we then find, in lowest-order perturbation theory,

$$< A_i^a(k)\, A_j^b(k') > = \frac{1}{(2\pi)^4}\,\delta^{ab}\delta(k+k')\, S_{im}(k)\, S_{jm}(k').$$

However, we have

$$S_{im}(k)\, S_{jm}(-k) = D_{ij}(k)\,,$$

so that $(2\pi)^{-4}\delta^{ab}D_{ij}(k)$ is a free propagator, in agreement with the result that follows from (1). Analogously, we find the following expression for the ternary correlation function:

$$< A_i^a(k_1)\, A_j^b(k_2)\, A_k^c(k_3) > = \frac{1}{(2\pi)^8}\,\delta(k_1+k_2+k_3)(-\alpha\,\epsilon_{lnp})$$

$$\times\; g f^{abc} S_{il}(k_1)\, D_{nj}(k_2)\, D_{pk}(k_3) + \text{cyclic}\,\begin{pmatrix}1 & 2 & 3\\ i & j & k\end{pmatrix}.$$

It can be shown with the help of the condition $\omega_1 + \omega_2 + \omega_3 = 0$ that the part of this expression which is proportional to α is identical to a self-doubling and is therefore zero. Now; using

$$\epsilon_{lmi} k_m/(\mathbf{k}^2+\omega^2) = \epsilon_{lmn} k_m D_{in}(k),$$

we find

$$< A_i^a(k_1)\, A_j^b(k_2) A_k^c(k_3) >$$

$$= \frac{1}{(2\pi)^{12}}\;\delta(k_1+k_2+k_3) D_{kk'}(k_3) D_{ii'}(k_1) D_{jj'}(k_2)\Gamma^{abc}_{i'j'k'}(k_1, k_2, k_3),$$

where

$$\Gamma^{abc}_{ijl}(k,p,q) = -\,i g f^{abc}\,(2\pi)^4\,(\delta_{ij}k_l + \delta_{jl}p_i + \delta_{li}q_j),$$

in agreement with the result found from (1). It is also a simple matter to show that the diffusion is constructed in such a manner that we have $\langle \nabla_i \dot{A}_i \rangle = 0$.

4. In summary, at the cost of losing the explicit relativistic invariance we have managed to formally write the quantum fluctuations in $YM(R^4)$ as the response of a *quadratically* nonlinear differential equation of *first* order to white noise. The next necessary step is to test the equivalence of the renormalized perturbation theories in this approach and in the standard formulation of $YM(R^4)$. Another interesting question is whether it would be possible to describe a "spontaneous transition to a stochastic state" in relativistically invariant gauges. Finally, with an infinite number of colors a white noise can be realized by the "frozen-momentum" procedure,[5] so that there is the possibility of carrying out a "freezing" program in Eq. (6).

I thank V. Ya. Fainberg for detailed discussions of this study and for useful comments. I also thank M. Vodzitskiĭ and N. S. Maslova for bringing Refs. 4 and 3, respectively, to my attention and for discussions of those papers.

[1)]Recalling that the theory is derived through a continuation from Minkowskiĭ space, we have $R^4 \partial x = (\mathbf{x}, x^4)$, where x^4 is the Euclidean time t; below we will use a dot to represent differentiation with respect to this time. The orientation of the space R^4 is chosen such that $\epsilon_{1234} = 1$.

[2)]When Eqs. (2) are rewritten in the Lorentz- and gauge-covariant form $1/2\, \bar{\eta}_{i\mu\nu} F_{\mu\nu} = \eta_i$, where $\bar{\eta}_{i\mu\nu}$ are the 't Hooft matrices,[2] they generate Yang-Mills equations with a current $I_\mu = \bar{\eta}_{i\lambda\mu} \nabla_\lambda \eta_i$. Here $\nabla_\mu I_\mu = \bar{\eta}_{i\lambda\mu}(1/2)[F_{\mu\lambda}\, \eta_i] = -[\eta_i, \eta_i] = 0$.

[3)]The analog of this formula for the Itô calculus is nontrivial; see Refs. 4 and 3.

[1]S. S. Chern, Complex Manifolds without Potential Theory, Springer, Berlin (1979).
[2]G. 't Hooft, Phys. Rev. D **14**, 3432 (1976).
[3]K. L. C. Hunt and J. Ross, J. Chem. Phys. **75**, 976 (1981).
[4]P. A. Meyer, "Geometrie differentielle stochastique sans larmes," in: Lecture Notes in mathematics, 851 (1981); "A differential geometric formalism for the Itô calculus," in: Lecture Notes in Mathematics, 850 (1981).
[5]J. Greensite and M. B. Halpern, Nucl. Phys. **B211**, 343 (1983).

Translated by Dave Parsons
Edited by S. J. Amoretty