Spinor structure of superspaces

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A realization of the $OSp(1,4)$ superalgebra is analyzed on the basis of canonically conjugate spinor variables with the opposite Grassman gauge. The relationship between the canonical spinor variables and the coordinates x^μ and θ^α of a de Sitter superspace is derived.

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The idea that the geometric properties of space and time are consequences of spinors, which are extremely simple structural elements, has been expressed in several places in the literature (see Ref. 1, for example, and, with reference to superspaces, Refs. 2 and 3). In this letter we adopt the example of a de Sitter superspace with the Osp (1,4) transformation supergroup to see how the properties of superspaces can be determined through the use of spinor variables.

We introduce the two real four-dimensional spinors η^α and y_α, which have anticommuting and commuting components, respectively, and we adopt the following definition of the Poisson brackets for functions of these spinors (see Ref. 4, for example):

$$\{A,B\} = A\,\frac{\overleftarrow{\partial}}{\partial y_\alpha}\frac{\overrightarrow{\partial}}{\partial \eta^\alpha}B - A\,\frac{\overleftarrow{\partial}}{\partial \eta^\alpha}\frac{\overrightarrow{\partial}}{\partial y_\alpha}B. \tag{1}$$

From (1) we have the gauge relations

$$\{A,B\} = C, \qquad a+b=c+1, \tag{2}$$

the commutation relations

$$\{A,\ B\ \} = -(-1)^{(a+1)(b+1)}\ \{B,A\ \}\ , \tag{3}$$

and the Jacobi identities

$$(-1)^{(a+1)(c+1)}\{A,\ \{B,C\}\ \} + (-1)^{(b+1)(a+1)}$$

$$\times\{B,\ \{C,A\}\ \} + (-1)^{(c+1)(b+1)}\{C,\ \{A,B\}\} = 0. \tag{4}$$

For the variables η^α and y_β directly, the brackets (1) become $\{\,y_\beta,\eta^\alpha\} = \delta^\alpha_\beta$. In relations (2)–(4), a, b, and c represent the Grassman gauge of A, B, and C, respectively. If we transform from the Grassman gauge of A, B, and C to the new gauge $\tilde{a} = a+1$, $\tilde{b} = b+1$, and $\tilde{c} = c+1$, then in this new "physical" gauge relations (2)–(4) have the form of the relations for a Lie superalgebra.

With respect to the Poisson brackets (1), power functions of the spinors η^α and y_α form an infinite-dimensional Lie superalgebra.

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Depending on whether the number of y_α factors in the power functions is odd or even, their physical gauge corresponds to the correct or incorrect relationship between the spin of the functions and the form of the commutation relations for them. It is easy to see that functions with the correct coupling of the spin with the statistics (we will call these "Pauli functions") form a closed Lie superalgebra. Functions of the type $y_\alpha f(\eta)$ are Pauli functions. Such functions form a closed and finite superalgebra. The 32 even elements of this superalgebra contain the superalgebra $GL(4,R)$ and 16 translations. The 32 odd elements are the generators of the supertransformations themselves.

The superalgebra of the functions $y_\alpha f(\eta)$ contains many different subalgebras and their nonequivalent representations, which are of interest from the standpoint of relativistic supersymmetry theory. Some particular superalgebras are (a) the superalgebra $SL(4,1/R)$ with the generators $iy_\alpha\eta^\beta$, y_α, $i\eta^\alpha(\eta y)$ ($(\eta y)\equiv \eta^\alpha y_\alpha$), which is a supergeneralization of the algebra $O(3,3)$ [with respect to the $O(3,3)$ group, η^α and y_α are two nonequivalent Majorana-Weyl spinors], and (b) the superalgebra $Osp(1,4)$, which is found from $SL(4,1/R)$ by imposing the additional condition of invariance of the bilinear antisymmetric form $C_{\alpha\beta}\, y^\alpha y'^\beta$ with respect to even generators.

The generators of the $Osp(1,4)$ superalgebra are $\frac{i}{2}(y_\alpha\eta_\beta + y_\beta\eta_\alpha)$, $y_\alpha + i\kappa\eta_\alpha(y\eta)$; in the course of determining these generators, we used the form $C_{\alpha\beta}$ in order to drop the spinor indices. In terms of the standard notation, the generators of the $Osp(1,4)$ superalgebra can be written

$$P_\mu = i\kappa\,\bar{\eta}\gamma_\mu y, \qquad M_{\mu\nu} = i\bar{\eta}\,\sigma_{\mu\nu} y, \tag{5}$$

$$Q_\alpha = y_\alpha + i\kappa\,\eta_\alpha(\bar{y}\eta)\,. \tag{6}$$

In the limit in which the parameter κ approaches zero, we go from (5), (6) to a Poincaré superalgebra.

We turn now to the question of how we can use the operator representation (5), (6) to relate the variables η_α and y_α to the ordinary coordinates of a de Sitter superspace, x^μ and θ^α.

It follows from (6) and (1) that for transformations by the generators Q_α we have

$$\delta\eta^\alpha = \{(a, Q), \eta^\alpha\} = a^\alpha - i\kappa\,(a\eta)\eta^\alpha \tag{7}$$

and

$$\delta y_\alpha = \{(aQ), y_\alpha\} = i\kappa\,[a_\alpha(\eta y) + y_\alpha(a\eta)]\,. \tag{8}$$

If we go from y_α to $\tilde{y}_\alpha(y,\eta)$,

$$\tilde{y}_\alpha = y_\alpha + \frac{i\kappa}{2}[\eta_\alpha(y\eta) - y_\alpha(\eta^2)] + \frac{5}{32}\kappa^2 y_\alpha(\eta^2)^2, \tag{9}$$

we find

$$\delta\tilde{y}_\alpha = if_{\alpha\beta}\tilde{y}^\beta, \tag{10}$$

where

$$f_{\alpha\beta}=f_{\beta\alpha}=-\frac{\kappa}{2}(a_\alpha\eta_\beta+a_\beta\eta_\alpha)\left(1+\frac{i\kappa}{4}\eta^2\right).$$

Transformations (7) correspond to coordinate transformations of a uniform space, $Osp(1,4)/Sp(4)$. The variables η^α can thus be identified with the Grassman variables θ^α, which are ordinarily used:

$$\theta^\alpha=\eta^\alpha. \tag{11}$$

To relate the variables y_α to the coordinates x^μ we note that relations (10) determine the transformation of a representation of the $Osp(1,4)$ group induced by a spinor representation of the $Sp(4)$ group.

Since the supertransformations of the coordinates x^μ are determined through a representation of the $Osp(1,4)$ group induced by a five-dimensional vector representation of the SO(3,2) group,[5] and since the latter representation is equivalent to an antisymmetric tensor representation of the $Sp(4)$ group, we can determine the relationship between y_α and x^μ by using

$$\tilde{y}_\alpha(y,\eta)=\kappa\, v_{\alpha\beta}(x)\tilde{y}^\beta(y,\eta), \tag{12}$$

where $v_{\alpha\beta}(x)=-v_{\beta\alpha}(x)$, $v^\alpha{}_\alpha(x)=0$, and, as a consequence of (12), $v^{\alpha\beta}(x)v_{\beta\alpha}(x)=-4/\kappa^2$.

In the presence of an internal symmetry, relation (12) completely determines the dependence of the coordinate $v_{\alpha\beta}$ on the variables $y_{\alpha i}$, η_i^α $(i=1,\ldots,N)$.

It follows from (11) and (12) that the coordinates θ^α and x^μ are functions of an anti-Pauli type with respect to Poisson brackets (1). We might also note that the Poisson brackets of θ^α and x^μ are not zero.

These properties of x^μ and θ^α suggest that x^μ and θ^α are not physical operators, in contrast with P_μ, M_X, and Q_α.

The basic and governing property of the coordinates x^μ and θ^α is their transformation law under transformations of the corresponding supergroups. In the functional dependences of the various quantities on x^μ and θ^α, on the other hand, only their Grassman gauge (the same as the usual gauge) is pertinent.

It is not clear at this point how to transform from the classical brackets (1) to quantum Poisson brackets.

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