Steady-state self-focusing of whistlers

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The self-focusing of whistlers has been studied numerically with steady-state boundary conditions. The process is halted in the final stage by the emission of a collapsing mode into a divergent mode.

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The self-focusing of whistlers has attracted interest, in particular, because of the possibility of some corresponding active experiments in the magnetospheric plasma. The problem is of considerable theoretical interest, since the self-focusing of whistlers has some very unusual features, as we will see below.

The results which has been carried out in laboratory experiments[1–3] are contradictory. A theoretical analysis[4,5] has shown that wave leakage from a waveguide formed during the contraction of a beam may play an important role in the self-focusing of whistlers.

In this letter we report a numerical study in which the simplifying assumptions of Refs. 4 and 5 were not used. It was possible to follow the self-focusing to its final stage in these numerical calculations. The results clearly demonstrate a leakage which results in an end to the self-focusing.

We consider the following problem. An axisymmetric right-hand polarized wave is incident on the boundary of a plasma ($z = 0$) in the direction parallel to the magnetic field, which is oriented along the z axis. We write the electric field in the plasma as

$$\vec{\mathcal{E}} = \frac{1}{2} [\, \mathbf{E}(r, z) \exp(ikz - i\omega t) + \text{c.c.}\,]. \tag{1}$$

Here $k = \omega_p \omega^{1/2}/c(\omega_c - \omega)^{1/2}$ is the wave number of the whistler, which is propagating along the z axis, and ω_p and ω_c are respectively the electron plasma and electron cyclotron frequencies. We work from Maxwell's equations for a cold plasma and from the MHD equations, supplemented with the ponderomotive force of the wave. We introduce the relative density variation $\nu = (N - N_0)/N_0$ and the relative variation of the external magnetic field, $\mathbf{b} = (\mathbf{B} - \mathbf{B}_0)/B_0$($N_0$ and $\mathbf{B}_0$ are the density and the magnetic field at $r = \infty$). We choose the initial field amplitude to satisfy $\nu \ll 1$ and $|\mathbf{b}| \ll 1$ at $z = 0$. We will see below that this situation hold for arbitrary z (because of the leakage, the field amplitude is quite low throughout the process). We can then linearize the MHD equations. In the steady state, they reduce to $\mathbf{v} = 0$ and

$$\beta \nabla \nu + [\, \mathbf{b}_0,\ \operatorname{rot} \mathbf{b}] = 4\pi \mathbf{f}/B_0^2\,, \tag{2}$$

where we are assuming $\beta \equiv 4\pi N_0 T / B_0^2 \ll 1$, and where f is the ponderomotive force of the wave.

We introduce the dimensionless variables

$$\rho = kr,\ \zeta = kz,\ \mathbf{E}' = \mathbf{E}/E_1,$$

$$E_1^2 = 8u\,(1 - u) B_0^2 \beta \omega_c^2/\,\omega_p^2, \tag{3}$$

where $u = \omega/\omega_c$. From (2) and the expression[6] for $\mathbf{f}$ we have $|\mathbf{b}| \sim \beta^{1/2} \nu \ll \nu$ and

$$\nu = |\, V\,|^2 - \frac{1+u}{1-u}(1-2u)^2 |\, V - U|^2 - \frac{u^2}{1-u} \,|\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho(V + \frac{1-2u}{2u}U)|^2, \tag{4}$$

where

$$V = E_r' - iE_\phi',\quad U = V - (1-u)(E_r' + iE_\phi')/\,(1+u)(1-2u), \tag{5}$$

and E_r' and E_ϕ' are components of the field $\mathbf{E}'$ in a cylindrical coordinate system.

Under the condition $\nu \ll 1$, Maxwell's equations can be put in the form

$$(i\frac{\partial}{\partial\zeta} + \frac{1}{2}\frac{\partial^2}{\partial\zeta^2})\,(b_{11}V + b_{12}U) + LV = B_{11}(\nu)\,V + B_{12}(\nu)U,$$

$$b_{21}(i\frac{\partial}{\partial\zeta} + \frac{1}{2}\frac{\partial^2}{\partial\zeta^2})\,V\ +\ LU = B_{21}(\nu)\,V,$$

$$L = \partial^2/\,\partial\rho^2 + (1/\rho)\,\partial/\partial\rho - 1/\rho^2, \tag{6}$$

$$b_{11} = -2(1-2u^2)/u^2,\ \ b_{12} = (1+u)(1-2u)/\,u^2,\ \ b_{21} = 4(1-u)/\,(1-2u),$$

$$B_{11} = -(1-2u)/u^2 + 2(1-u)\nu/u, \quad B_{21} = -2(1-u)\nu/(1-2u),$$

$$B_{12} = (1-2u)/u^2 + (1-u)(1-2u)\nu/2u^2.$$

The quantity U in (6) has the following meaning: The wave emerging from the waveguide (this wave is excited as the result of a tunnel conversion[7] of the self-focusing wave) has the following form[5] at $r \to \infty$:

$$r^{-1/2}\,\mathbf{E}^0 \exp\{i(k_\perp r + kz - \omega t)\}, \tag{7}$$

where ω and k are the same as in (1), and

$$k_\perp = k\,(1-2u)^{1/2}/u, \quad E_r^0 + iE_\phi^0 = (1+u)(1-2u)(E_r^0 - iE_\phi^0)/(1-u). \tag{8}$$

Working from (8), we easily see that U, in contrast with V [see (5)], must rapidly vanish in the limit $r \to \infty$. Furthermore, we can expect U to be a smoother function of r and z than V, since V is a superposition of two waves—the self-focusing wave and the wave resulting from the tunnel conversion—while the field of the latter wave is largely subtracted from U. This assumption is supported by the numerical results (discussed below). We can accordingly ignore the term with $\partial^2 U/\partial\xi^2$ in the first of Eqs. (6); as a result, system (6) simplifies substantially, taking a form convenient for numerical solution.

If the transverse dimension of the beam is significantly greater than k^{-1} and if $\nu \ll 1$, then $U \approx V$, and from (4) and (6) we find the Schrödinger equation[5]

$$i\frac{\partial V}{\partial \zeta} + \frac{1-2u}{4(1-u)} LV + \frac{1}{2}|V|^2 V = 0, \tag{9}$$

which describes the initial stage of the beam evolution.

System (4), (6), and Eq. (9) were solved numerically by an implicit difference scheme. The initial field of the beam (at $\xi = 0$) was specified to be

$$V = V_0(\rho/s)\exp(-\rho^2/2s^2). \tag{10}$$

In addition, for Eqs. (6) we specified $U|_{\zeta=0} = V|_{\zeta=0}$, $\partial V/\partial\zeta|_{\zeta=0} = 0$. The calculation region was $\zeta \geqslant 0$, $0 \leqslant \rho \leqslant \rho_{\max} = 4s$. At $\rho = 0$ we assumed $V = U = 0$. At the outer boundary, $\rho = \rho_{\max}$, we imposed the condition that the energy emerging from the calculation region was absorbed.

Let us examine the results. Self-focusing occurs when V_0^2 exceeds a certain critical value (0.007 for $s = 21.5$ and $u = 0.3$) In the first stage of the self-focusing, a tubular wave structure forms and contracts toward the beam axis. For this structure, there is a smooth change in V, and we have $U \approx V$ and $\nu > 0$. In this stage, the solution of system (4), (6) is approximately equal to the solution of Eq. (9). The end of the first stage corresponds approximately to Fig. 1b, where $|V|^2_{\max} = |V(\rho_0)|^2 \approx 0.09$ for $\rho_0 \approx 11$, $|U|^2_{\max} \approx 0.07$ and $\nu_{\max} \approx 0.09$. When the transverse dimension of the beam becomes comparable to the longitudinal wavelength of the whistler, the second stage begins. At this point the contraction comes to a halt because of the intense tunnel conversion of the wave trapped in the waveguide into an outgoing wave with the

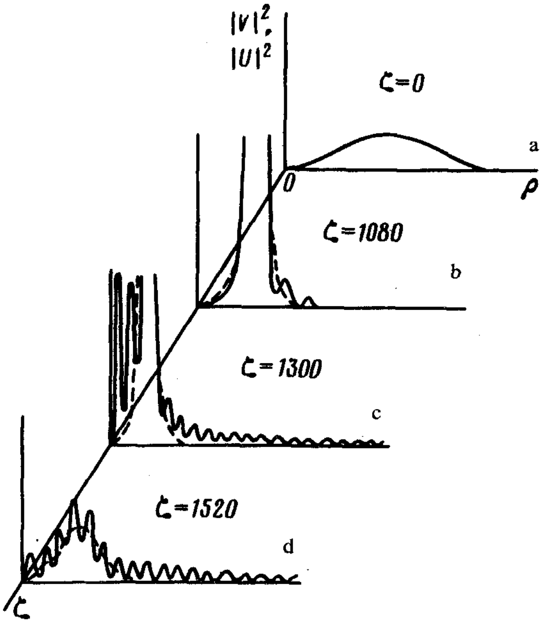

FIG. 1. Evolution of a beam of whistlers for $u=0.3$, $V_0^2=0.013$, and $s=21.5$. Solid curves—$|V|^2$; dashed curves—$|U|^2$.

asymptotic form (7). This situation can be seen clearly in Fig. 1c, where we have $|V|^2_{max}=0.12, \rho_0 \approx 6$, $|U|^2_{max} \approx 0.07$, and $\nu_{max} \approx 0.1$. The energy leakage subsequently becomes so rapid that $|V|^2_{max}$ begins to decrease, and the beam progressively loses its energy. In Fig. 1d, for example, we have $|V|^2_{max} \approx 0.02$, $\rho_0 \approx 9$, $|U|^2_{max} \approx 0.013$ and $\nu_{max} \approx 0.02$. It can be seen from Fig. 1 that the function $U(\rho,\xi)$ is much smoother than $V(\rho,\zeta)$ and has a width on the order of that of the waveguide. As follows from the definition of U, this circumstance is the primary reason why the oscillations in Fig. 1 are linked with the leakage.

In the second stage, the solution of system (4), (6) is naturally quite different from the solution of Eq. (9), which does not incorporate the leakage effect. This situation is illustrated by Fig. 2.

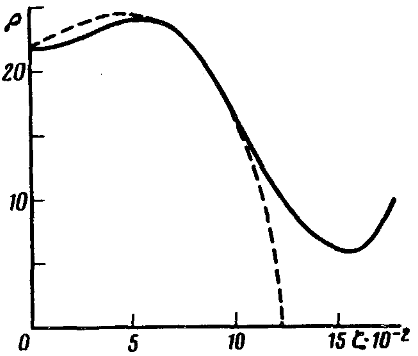

FIG. 2. Solid curve—Position of the field maximum $|U|^2$; dashed curve—position of the field maximum $|V|^2$ in Eq. (9), for the parameters of Fig. 1.

We thus see that the self-focusing is stopped by the energy loss caused by the conversion of the wave into the mode described by Eqs. (8).

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