Stochastic aspect of two-particle scattering

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A stochastic aspect of classical two-particle scattering is demonstrated in the case of the scattering of vortex pairs in two-dimensional hydrodynamics.

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1. The two-particle scattering problem, which might seem quite familiar, can occasionally demonstrate some totally unexpected properties. Unusual effects will of course, occur if the colliding particles have internal structure, as, for example, when the scattered entities are bound states of their elementary constituents. A case of particular interest is that in which these elementary constituents cannot exist in their "bare" form.

Apparently the simplest example of this type, with the minimum number of degrees of freedom, is a system of point vortices in the two-dimensional hydrodynamics of an ideal incompressible fluid. A vortex dipole—a set of two vortices with circulations equal in modulus and opposite in sign—should be treated as a "free" particle in this system. This choice is imposed by the following circumstance.

A system of N vortices with circulations κ_i $(i = 1,...,N)$ is a Hamiltonian system whose phase space consists of the set of Cartesian coordinates of the vortices, x_i, y_i $(i = 1,...,N)$. The Poisson brackets are given by

$$\{A, B\} = \Sigma \kappa_i^{-1}\left(\frac{\partial A}{\partial x_i}\frac{\partial B}{\partial y_i} - \frac{\partial A}{\partial y_i}\frac{\partial B}{\partial x_i}\right), \tag{1}$$

and the Hamiltonian representing the regularized kinetic energy of the liquid is

$$H = {}^1/_2 \sum_{i<j} \kappa_i \kappa_j \ln\left[(x_i - x_j)^2 + (y_i - y_j)^2\right], \tag{2}$$

so that the equations of motion are

$$\dot{x}_i = \{x_i, H\}, \quad \dot{y}_i = \{y_i, H\}. \tag{3}$$

It follows from the translational invariance of H that the momenum $\mathbf{P}$ is conserved,

$$P_x = \Sigma \kappa_i y_i, \quad P_y = -\Sigma \kappa_i x_i,$$

so that we have $\{P_x, H\} = 0$, $\{P_y, H\} = 0$. However, we have

$$\{P_x, P_y\} = \Sigma \kappa_i, \tag{4}$$

and we can say in quantum-mechanical terms that the system can have a definite momentum value (and only such an entity could be called a "particle") only if its total circulation vanished: $\Sigma\kappa_i = 0$.

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The simplest entity of this type is a vortex dipole which has a constant velocity and which is moving along a straight line. We wish to emphasize that an isolated vortex, like any set of vortices with $\Sigma\kappa_i \neq 0$, cannot be asymptotically free.

We will draw a distinction between a vortex dipole with a zero resultant intensity (an elementary particle) and a vortex pair with an arbitrary resultant intensity, which is not a particle.

2. A vortex dipole of dimension a has a momentum $\mathbf{P} = \kappa[\mathbf{n}\times\mathbf{a}]$, where $\mathbf{a}$ is a vector directed from the $-\kappa$ vortex to the $+\kappa$ vortex, and $\mathbf{n}$ is the unit normal to the plane. The dipole has an energy $\epsilon = (\kappa^2/2)\ln(P^2/\kappa_2)$ and is moving at a velocity $\mathbf{v} = \partial\epsilon/\partial\mathbf{P} = \kappa^2\mathbf{P}/P^2$, so that the position of the center of the dipole is

$$\mathbf{R}(t) = \mathbf{v}t + \mathbf{R}_0.$$

We consider a collision of two dipoles with circulations κ_1, κ_2, $|\kappa_1 - \kappa_2| \ll \kappa_{1,2}$, and with momenta $\mathbf{P}_1^{\text{in}}$ and $\mathbf{P}_2^{\text{in}}$ at infinity; we assume $a_1 \sim a_2 \sim a$. If the impact parameter of the collision, $\delta = \min_t |\mathbf{R}_1(t) - \mathbf{R}_2(t)|$ (where the motion of the dipoles is assumed free), is large, satisfying $\delta \gg a$, the scattering of the dipoles is not qualitatively different from ordinary two-particle scattering. The behavior of the scattering angle is described by $\phi_{\text{sc}} \sim (a^2/\delta^2)$ (Fig. 1a). At $\delta \sim a$, however, we find a qualitatively new behavior: dipoles closing on each other close to a distance on the order of their original dimensions. The dipoles may split into two vortex pairs with the sets of circulations $(\kappa_1, -\kappa_2)$ and $(-\kappa_1, \kappa_2)$. The dimensions of the pairs, l_1, l_2, are of the same order as those of the dipoles, a_1, a_2. The pairs are not particles: The resultant circulations in them, $\Delta\kappa = |\kappa_1 - \kappa_2|$, are not zero. The pairs thus cannot go off to infinity. They revolve around a circle of radius $\rho \sim \kappa l/\Delta\kappa$ at an angular frequency $\omega \sim \Delta\kappa/l^2$ in opposite directions. Since $\Delta\kappa \ll \kappa$, we have $\rho \gg l$, and for the greater part of the time the pairs interact only slightly. Since their trajectories intersect, however, the pairs close to distances on the order of their own dimensions from time to time. As a result of this "strong interaction," the vortices making up the pairs may either return to their original partner (in this case vortex dipoles form and go off to infinity, and the scattering process is thus terminated) or be scattered through an angle on the order of unity, with the process continuing until the next "strong interaction," etc.

Even from this qualitative picture it can easily be seen that the scattering of vortex dipoles is definitely not a trivial question. In the first place, even at the classical level this scattering is of a clearly defined resonant nature and is accompanied by the

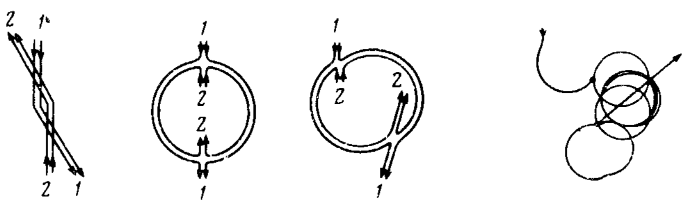

FIG. 1. Trajectories of vortices for various values of the impact parameter. a—$\delta = 2$; b—$\delta = 0$; c—$\delta = 0.5$; d—$\delta = 0.7897$. The trajectory of a single vortex is shown.

creation of an intermediate long-lived (resonant) state. Second, scattering characteristics such as the scattering angle φ_{sc} (the angle between $\mathbf{P}_1^{in}$ and $\mathbf{P}_1^{out}$) and the time spent in the resonant state (T_{res}) are extremely singular functions of the collision parameters (i.e., of the impact parameter and of the angle between $\mathbf{P}_1^{in}$ and $\mathbf{P}_2^{in}$).

3. To illustrate the discussion above we report some data on the scattering of dipoles found through a numerical integration of system (3). We consider the simplest case: the collision of antiparallel dipoles with $\mathbf{a}_1 = -\mathbf{a}_2$ (the original velocities are also antiparallel), as a function of the impact parameter δ. We set $\mathbf{a}_2 = \mathbf{a}$, $|\mathbf{a}| = 1$, $\kappa_1 = \kappa = 1$, $\kappa_2 = \kappa + \Delta\kappa$, and $\Delta\kappa = 0, 1$. (For a qualitative observation of these effects it is necessary to satisfy $\Delta\kappa \ll \kappa$.)

Figure 1b shows the trajectories of vortices in a head-on collision of dipoles with $\delta = 0$. At small values $\delta \ll a$ the scattering angle is $\varphi_{sc} \sim \delta / a$. In this case the dipoles split into pairs as they close on each other. After the pairs trace out a semicircle, they convert back into dipoles (see Fig. 1c, for example). When δ reaches the threshold value $\delta_c \simeq 0.785...$, however, the first collision of the pairs does not result in their conversion into dipoles; the pairs continue to move along circles until the next collision, etc. (Fig. 1d). As δ goes through its critical value, the function $\varphi_{sc}(\delta)$ is discontinuous, and $T_{res}(\delta)$ changes abruptly. Figure 2 shows $T_{res}(\delta)$ and $\cos\varphi_{sc}(\delta)$; we can clearly see the jumps in the function T_{res} and (especially) $\cos\varphi_{sc}$. At certain values, δ_* (indicated by the arrows) we have $T_{res} \to \infty$. This result means that two dipoles can capture each other and can spend an unbounded time at distances not exceeding the diameter of the resonant circles. The capture state is unstable: It is disrupted upon a

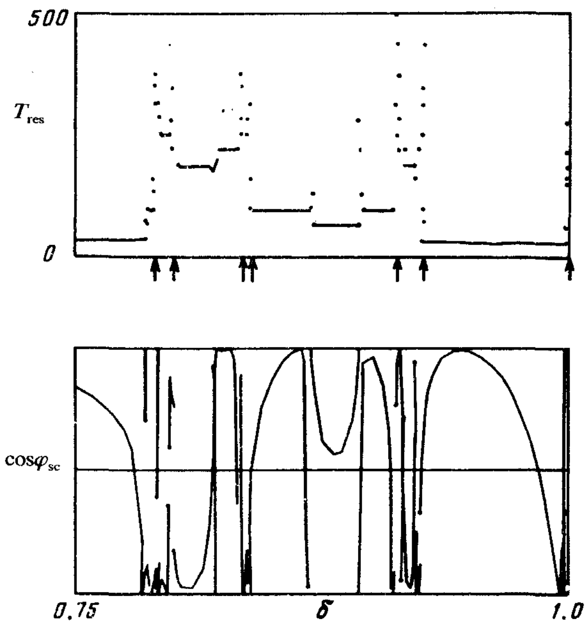

FIG. 2. Dependence of $T_{res}(\delta)$ and $\cos\varphi_{sc}(\delta)$ on the impact parameter δ. The function $\cos\varphi_{sc}$ is chosen for clarity.

small change in the parameters, i.e., the pairs convert into dipoles and then go off to infinity.

4. The dynamics of the four-vortex system has been studied by various investigators.[1–4] Ziglin[2] proved that there is no additional integral of motion, so that the system is not integrable. On the other hand, Refs. 1–4 all dealt with a system of vortices of the same sign. In this case the accessible region in phase space is compact, and the manifestations of the stochastic behavior are suppressed because of the invariant separating tori.

Our formulation of the problem seems preferable for two reasons. First, as shown in Sec. 1, only a system with a zero resultant intensity if physically meaningful. Second, this formulation not only helps explain the mechanism for the appearance of the stochastic behavior but also helps us see a definitely unusual pattern in the scattering of two composite particles. We have assumed in this model that the radii of the vortex cores are much smaller than the dimensions of the dipoles. In this case we can ignore the change in the circulation of the vortices during the collision (we recall that the resultant circulation of the system is conserved). Taking the infinite radii of the vortex cores into account does not change any of the qualitative results (the structure in the scattering-angle function and the resonant states). As for the capture state, we note that the redistribution of the circulations between the vortices in the course of the collision has the consequence that all the circulations become different in modulus. As a result, dipoles cannot form in subsequent collisions, and the capture state becomes stable. We wish to thank É. I. Rashba for useful discussions and Ya. G. Sinaĭ for constant interest in this study.

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