Isotropization of arbitrary cosmological expansion given an effective cosmological constant

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It is shown that given an effective cosmological constant it is possible to construct the asymptotic structure of inhomogeneous cosmological expansion. This structure would contain the maximum possible number of arbitrary functions of three coordinates and would describe exponentially rapid local isotropization of the universe.

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One of the most important problems in cosmology is explaining why the visible part of the universe $\sim 10^{28}$ cm in length is homogeneous and isotropic to such an extent. It is well known that among the solutions of Einstein's equations without a cosmological constant the homogeneous isotropic solution of Friedmann is a very separate, particular case. Even in the narrow class of homogeneous models, isotropization in the general sense of the word, i.e., equality of expansion velocities (Hubble constants) in all directions correct up to small corrections, is attained for finite t only under special assumptions about the initial conditions, while asymptotic isotropization with $t \to \infty$ is possible only in type I, V, and VII Bianci models.[1–3] Going over to even more general inhomogeneous models makes the situation even worse. In particular, the inhomogeneous quasi-isotropic solution of Lifshitz-Khalatnikov[4] is close to the isotropic solution only near the singularity ($t \to 0$), while for $t \to \infty$ it generally becomes anisotropic and inhomogeneous. Thus, within the framework of Einstein's equations without a cosmological term, the isotropy of the universe must be postulated, starting from observational data.

In this paper it is shown that the situation fundamentally changes if the energy-momentum tensor of matter contains a positive cosmological term $T_i^k = \epsilon_V \delta_i^k$, where $\epsilon_V > 0$ is the energy density of the vacuum. Below it is not important whether the cosmological constant is a true constant ($\epsilon_V \equiv$const) or is only an effective constant (then $\epsilon_V \approx$ const on the strength of the equations of motion over a certain time interval τ). The difference lies only in the fact that in the second case the asymptotic expression (2) will be an intermediate expression, correct for $H^{-1} \ll t \ll \tau$, where $H^2 = (8\pi G \epsilon_V / 3)$ (it is assumed that $H\tau \gg 1$; the velocity of light $c = 1$). For applications, however, the second case is important, in which the universe passed through regime (2) at early stages of its evolution; in the first case, on the other hand, we obtain only the uninteresting prediction of local isotropization of the universe in the distant future. At present, two reasonable methods are known for obtaining an effective cosmological constant at early stages of the evolution of the universe: due to single-loop quantum gravitational effects[5] and due to a phase transition related to the scalar Higgs field.[6,7]

The nonsingular asymptotic solution of the equations

$$R_i^k - \frac{1}{2}\delta_i^k R = 8\pi G\epsilon_V\delta_i^k, \quad (1)$$

which we seek for $t\to\infty$ is similar to the quasi-isotropic solution,[4] but differs from it by the large (maximum) number of physically different, arbitrary functions and by the fact that it is an expansion near $t=\infty$, rather than at $t=0$. we have the series

$$ds^2 = dt^2 - \gamma_{\alpha\beta}dx^\alpha dx^\beta \quad ,$$

$$\gamma_{\alpha\beta} = e^{2Ht}a_{\alpha\beta} + b_{\alpha\beta} + e^{-Ht}c_{\alpha\beta} + ..., \quad (2)$$

where $a_{\alpha\beta}$, $b_{\alpha\beta}$, $c_{\alpha\beta}$ are functions of three spatial coordinates. The operation of lifting indices and covariant differentiation are performed below with the help of the time-independent metric $a_{\alpha\beta}$.

The tensor $b_{\alpha\beta}$ completely determines $a_{\alpha\beta}$

$$b_\alpha^\beta = \frac{1}{H^2}\left(\mathcal{P}_\alpha^\beta - \frac{1}{4}\mathcal{P}\delta_\alpha^\beta\right), \quad (3)$$

where $\mathcal{P}_{\alpha\beta}$ is the three-dimensional spatial curvature tensor, constructed according to the metric $a_{\alpha\beta}$. For c_α^β, four conditions follow from (1):

$$c \equiv c_\alpha^\alpha = 0, \quad c_{\alpha;\beta}^\beta = 0. \quad (4)$$

The next terms in the series (2) are proportional to integer positive powers of e^{-Ht} which are uniquely determined by $a_{\alpha\beta}$ and $c_{\alpha\beta}$.

The solution (2) contains four (maximum possible number) physical arbitrary functions of three coordinates and, therefore, is a general solution[4] which is stable relative to perturbations that are not too large.[1)] Two physically arbitrary functions are contained in $a_{\alpha\beta}$ and two are contained in $c_{\alpha\beta}$. Three functions in $a_{\alpha\beta}$ are eliminated by three transformations of spatial coordinates, not including time, and the fourth function is eliminated by the transformation $\tilde{t} = t + \varphi(x^\alpha)$, $\tilde{x}^\alpha = x^\alpha - (1/2H)(\partial\varphi/\partial x^\beta)a^{\alpha\beta}e^{-2Ht} + ...$, which does not destroy synchronism, with which the transformed metric again has the form (2) with $\tilde{a}_{\alpha\beta} = a_{\alpha\beta}e^{-2H\varphi}$.

Matter with the equation of state $p = k\epsilon$ ($k=$ const,$0\leqslant k<1$) can be smoothly included in (2) with the required number of additional physically arbitrary functions. For $k=0$ (at the early stages of evolution of the universe this could be heavy metastable particles or primeval black holes) instead of (4) we have

$$\epsilon = -\frac{3H^2}{8\pi G}e^{-3Ht}c; \quad c<0; \quad (5)$$

$$u_\alpha = \frac{1}{2Hc}(c_{\alpha;\beta}^\beta - c_{,\alpha}); \quad u_0 \to 1.$$

The velocity u_α is generally a vortical velocity. For $k>0$ instead of (4) we have

$$c = 0; \quad \epsilon u_0 u_\alpha = -\frac{3H}{16(1+k)\pi G} e^{-3Ht} c^{\beta}_{\alpha;\beta} . \tag{6}$$

If $0<k<1/3$, then $\epsilon \propto \exp(-3(1+k)Ht), u_0 \to 1, u_\alpha \propto \exp(3kHt)$; $k=1/3$: $\epsilon \propto \exp(-4Ht)$; for $u_0 \to \text{const} \neq 1, u_\alpha \propto e^{Ht}$ for $(1/3)<k<1: \epsilon \propto \exp(-[2(1+k)/(1-k)]Ht)$, $u_0 \to \infty$, $u_\alpha \propto \exp([2k/(1-k)Ht]$.

We can see that rapid local isotropization with expansion over characteristic time $\Delta t \sim H^{-1}$ is a typical phenomenon in the presence of a cosmological constant $\Lambda = 3H^2$. In this case, $C_{iklm} C^{iklm} \propto e^{-6Ht} \to 0$ for $t \to \infty$, where C_{iklm} is the Weyl tensor. All inhomogeneous perturbations, except the so-called nonsingular mode of gravitational waves, approach zero with $t \to \infty$, and the latter remains constant in amplitude, but its characteristic wavelength $\to \infty$. For this reason, space-time inside a constant physical volume rapidly approaches de Sitter's space-time, and the initial conditions are forgotten.

Thus the cosmological constant is the best "isotropizer": It is capable of eliminating or extending over very large scales all types of inhomogeneities. For comparison, we recall that the maximum effect that particle creation can have is to cause the expansion to reach the Lifshitz-Khalatnikov quasi-isotropic solution.[2)] After the decay of the effective cosmological constant and the end of the quasi–de Sitter state (2), perturbations begin to grow once again, but if stage (2) lasted for a sufficiently long time (in practice, it is sufficient to have $H\tau \sim 60$–70), then up to the present time the homogeneity and isotropy of the observed part of the universe, attained at stage (2), have not had sufficient time to break down. Thus, in the model of an intermediate quasi-de Sitter stage (2), we obtain a natural explanation of the approximate homogeneity and isotropy of the universe on a scale 10^{28} cm with only one assumption, viz., that the quantity $H\tau$ is sufficiently large.

We can say that intermediate stage (2) is an ordering, "antientropy" stage. Formally, there is no contradiction here with the second law of thermodynamics, since at stage (2) entropy does not disappear, but is diluted, spread out on very large scales. From a fundamental point of view, it is also interesting that the universe, having passed through stage (2) and appearing as locally isotropic, on very large scales is typically greatly inhomogeneous due to the dependence of $a_{\alpha\beta}$ on the spatial coordinates. From observations of the relict radiation it follows that the universe is approximately homogeneous and isotropic even on scales $\sim 10^{31}$ cm[10], but there are no reasons to expect that this ordering must remain for arbitrarily large scales. The characteristic size of the region of ordering is of the order of $H^{-1}\exp(H\tau)Z_H$, where Z_H is the red shift, corresponding to the end of stage (2) (in models constructed in Refs. 5–7, $Z_H \sim 10^{28}$–10^{31}).

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[1)] Of course, other general solutions of Eqs. (1) also exist, which concern different physical situations. Thus, near a singularity, the cosmological term is not significant and the oscillatory Belinskiĭ-Lifshitz-Khalatnikov regime occurs.[8] There is also a solution that describes a solitary black hole with small damped perturbations against the background formed by a de Sitter space.

[2] An example of a quasi-isotropic anisotropy, which does not disappear as a result of particle creation, was examined in Ref. 9.

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