Weakly bound states of an electron in an external electromagnetic field

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A model-independent method has been developed for calculating the spectra of weakly bound states of an electron with an arbitrary angular momentum l in an external electromagnetic field. Some results for specific fields are reported.

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1. We consider an electron with a low energy ($kr_c \ll 1$, $k=\sqrt{2E}$, $\hbar = m = e = 1$) in a potential $U(r)$ of radius r_c and in a uniform external field. The Schrödinger equation can be solved exactly at large distances ($r \gg r_c$) for this problem, while at $r \ll L$ the external field can be ignored, and the problem becomes spherically symmetric [$L=\epsilon^{-1/3}$ for an electric field and $L=(\mathscr{H}/c)^{-1/2}$ for a magnetic field]. The spectrum is found by requiring that the solutions join in the region

$$r_c \ll r \ll min\ \{ L, k^{-1} \} \tag{1}$$

(the condition $L \gg r_c$ presupposes that the field is not overly strong, but no condition is imposed on the relationship between L and k^{-1}).

We introduce $G_{lm}(\mathbf{r}, E)$, a complete system of solutions of the Schrödinger equa-

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tion with $U=0$ corresponding to decaying (or diverging) waves in the limit $r\to\infty$. We also require that in the limit $r\to 0$ the G_{lm} must contain singular terms of the type $r^{-l'-1}Y_{l'm}(\mathbf{n})$ only with $l'=l$, $m'=m$, so that in region (1) we have

$$G_{lm} \approx r^{-l-1} Y_{lm} + \sum_{l'm'} A_{lm}^{l'm'}(L,E) r^{l'} Y_{l'm'} . \quad (2)$$

These boundary conditions determine $A_{lm}^{l'm'}$ unambiguously.

We use (2) in the expansion $\psi = \Sigma c_{l'm'} G_{l'm'}$ in region (1), while if we come from small values of r we have the following in this region:

$$\psi_{in} = \Sigma \tilde{c}_{l'm'} Y_{l'm'} [r^{-l'-1} + B_{l'}(E) r^{l'}] ,$$
$$\frac{[(2l+1)!!]^2}{2l+1} B_l = k^{2l+1} \operatorname{ctg} \delta_l \approx -\frac{1}{a_l} + r_l E, \quad (3)$$

where δ_l, a_l, and r_l are the phase shift, the scattering length, and the effective radius in the field $U(r)$. If there is a shallow level with an angular momentum l in the potential $U(r)$, the scattering will be anomalously strong in this partial wave, the phase shifts $\delta_{l'\neq l}$ will be small, and we can discard the singular terms $r^{-l'-1}$ with $l'\neq l$. Joining ψ and ψ_{in} in region (1), we have $c_{lm} = \tilde{c}_{lm}$, and from the condition for the existence of a nontrivial solution we find an equation for $c_{lm'}$

$$\det (A_{lm}^{lm'}(L,E) - B_l(E)\,\delta_{mm'}) = 0 , \quad (4)$$

which determines the energy spectrum.

We will illustrate the method with some specific examples.

2. For a uniform magnetic field, conservation of l_z causes (4) to split into $2l+1$ independent equations. For $m=\pm l$ we have

$$G_{l,m=\mp l}(r,E) = [(\frac{\partial}{\partial x} + \omega x \mp i (\frac{\partial}{\partial y} + \omega y)]^l G_0(\mathbf{r},\mathbf{r}'=0,\tilde{E}), \quad (5)$$

where $\tilde{E} = E - (|m| + m)\omega$, $\omega = 1/2L^2$, and G_0 is the Green's function of the particle in a uniform magnetic field. Comparison of (2) with (5) yields A_{lm}^{lm}, and then we find from (4)

$$-\frac{1}{a_l} + r_l (E - m\omega)$$
$$= \frac{2^{2l+3/2}}{\sqrt{\pi}} \omega^{l+1/2} \int_0^\infty \frac{dt}{\sqrt{t}} \frac{d^{l+1}}{dt^{l+1}} \left\{ \left(\frac{t}{(1-e^{-2t})} \right)^{l+1} \exp [-(1 + |m| + m - \frac{E}{\omega}) t] \right\} . \quad (6)$$

In the case $|m| = l$, we note the magnetic field adds only states with $l' \geqslant l+2$, whose contribution to the energy is suppressed because of the small quantity $kr_c \ll 1$. Equation (6) thus describes the spectrum for arbitrary values of a_l. At $|a_l| \gtrsim L^{2l+1}$, (6) describes a change in the spectral structure. This question was discussed in detail in Ref. 1 for a square well[1)] with $l=0.1$. If $|a_l| \ll L^{2l+1}$ and $a_l < 0$, we find from (6) the equation for a perturbation theory in the length a_l:

$$E_{lm} - (|m| + m + 1)\omega \approx -2^{2l+1} [(2l+1)!!]^2 \omega^{2l+1} a_l^2 \quad (7)$$

(the bound state arises only if there is a magnetic field). If $r_c^{2l+1} \ll a_l \ll L^{2l+1}, a_l > 0$, and if there is a shallow level in $U(r)$ even in the absence of $\vec{H}$, we find from (6) the familiar perturbation-theory result for the paramagnetic and diamagnetic shifts of such a level (the scattering length should be renormalized because of the term in the Hamiltonian which is quadratic in $\mathscr{H}$; this renormalization is important only for $l \geqslant 2$). If $a_l \lesssim r_c^{2l+1}, a_l > 0$, there is no shallow level in the well, and no shallow level appears when a magnetic field is imposed.

Equation (6) determines the spectrum of shallow levels with $E_{lm} < (|m| + m + 1)\omega$. An analytic continuation of (6) along the energy scale yields the spectrum of quasidiscrete levels in the higher-order Landau bands.

3. For a particle in a circularly polarized wave, a transformation to a rotating coordinate system converts the Hamiltonian to

$$\hat{H} = -\frac{\Delta}{2} + U(r) - \omega l_z + \epsilon x \tag{8}$$

(ϵ and ω are the wave amplitude and frequency). If there is a shallow level with an angular momentum l in $U(r)$, Hamiltonian (8) has $2l+1$ quasidiscrete levels corresponding to this shallow level. For $l = 1$, Eq. (4) splits into two independent equations: one for $m = 0$ and one for the "entangled" values $m = \pm 1$. Here we have $G_{10}(\mathbf{r}, E) \propto \partial/\partial z G_0(\mathbf{r}, \mathbf{r}' = 0, E)$, where G_0 is the Green's function of operator (8) with $U(r) \equiv 0$ (Ref. 3). For $l = 1$, $m = 0$, Eq. (4) yields

$$-\frac{1}{a_1} + r_1 E_{10} = \frac{4}{i\sqrt{2\pi i}} \int_0^\infty \frac{dt}{\sqrt{t}} \frac{d^2}{dt^2} \exp\left(iE_{10}t + if(t)\right), \tag{9}$$

$$f(t) = \frac{\epsilon^2}{t\omega^4}(1 - \cos\omega t) - \frac{\epsilon^2}{2\omega^2}t\,.$$

In the lowest-order approximation in ϵ^2, we find from (9)

$$E_{10} \approx E_1^{(0)} + \frac{4\epsilon^2}{5\sqrt{2}\omega^4 r_1}\Big[-(\omega - E_1^{(0)})^{5/2} + 2(-E_1^{(0)})^{5/2}$$

$$+ \frac{15}{4}\omega^2(-E_1^{(0)})^{1/2} + i(\omega + E_1^{(0)})^{5/2}\Big], \tag{10}$$

where $E_1^{(0)} < 0$ is the energy of the level in the potential $U(r)$. If $\omega > |E_1^{(0)}|$, E_{10} acquires an imaginary part, which determines the level width due to single-photon ionization. For the case of n-photon ionization we find the level width from (9) ($r_l < 0$ for $l \geqslant 1$):

$$\Gamma_n \approx \frac{12\sqrt{2}}{(-r_1)} \frac{(n\omega - |E_1^{(0)}|)^{n+3/2}}{n!\,(2n+3)!!}\left(\frac{\epsilon^2}{\omega^4}\right)^n. \tag{11}$$

At $\omega \gg |E_1^{(0)}|$ we find from (10) $\mathrm{Re}E_{10} \approx \mathrm{Im}E_{10} \propto \omega^{-3/2}$. This behavior is disrupted at $\omega \gtrsim r_c^{-2}$ (the shift and width of the level at such frequencies are determined by the wave field at $r \lesssim r_c$), so that in our approach we must renormalize the scattering length and the effective radius to allow for the short-range effect of the field. In the limit $\omega \to 0$ the width is exponentially small. We can use (9) to calculate the corrections to the shift and width of the quasistationary state which arises in the static limit.

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[1)]See Ref. 2 and the literature cited there regarding the change in the spectral structure for a Coulomb potential with a short-range distortion.

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