Abelian dominance in a space-time lattice

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Orbits constructed from a Cartan subgroup of a gauge group are extreme in a space-time lattice. The relationship with the singular gauge in quantum chromodynamics is discussed.

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't Hooft[1] and Ezawa and Iwazaki[2] have predicted and discussed certain consequences of the hypothesis of the dominance of Abelian degrees of freedom in the long-range physics in quantum chromodynamics (at distances comparable to the confinement radius). Large distances have recently been studied quite successfully in quantum chromodynamics in gauge theories on a space-time lattice of finite size. Is Abelian dominance manifested on a space-time lattice?

In lattice calculations, the dynamic variables are the matrices U_{IJ} from the compact gauge group G which are assigned to each line of the lattice, while the quantites which are to be calculated are functions defined on group G. The action of a compact group on a compact manifold decomposes into orbits and layers constructed on the subgroups of the group G (Ref. 3). The number of layers is finite.[4] If the group G acts as

an internal automorphism, the set of orbits is the set of conjugate classes constructed on the subgroups $H_I \subset G$, and we have $M_I = \{gH_Ig^{-1}$, where g is any element of $G\}$. Distinctive among the orbits on the subgroups $H_I \subset G$ are those constructed on the Cartan subgroup $H_c \subset G$. Here H_c is the maximum Abelian subgroup of G; it is isomorphic to the product $\underbrace{\mathrm{U}(1) \otimes \mathrm{U}(1) \otimes \ldots \otimes \mathrm{U}(1)}_{r}$ of the group U(1), where r is the rank of group G. For the SU(n) groups we have $r = n - 1$. In this case the orbit is isolated in a layer.[3] We assume a point $p \in M_c$, and we assume that an orbit constructed on the Cartan subgroup H_c passes through this point. It can then be shown that the space tangent to the layer $S(p)$ and that tangent to the orbit $G(p)$ at the point $p \in M_c$ coincide: $T_p[S(p)] = T_p[G(p)]$. For orbits constructed from other subgroups $H_I \subset G$, we might note, we have $T_p(S(p)) = T_p(G(p)) \oplus L(p)$, where G, an invariant vector function on the manifold M_I with the Cartan-Killing metric, maps M_I onto the space tangent to M_I, $T(M_I)$. The vector field is tangent to the layer $S(p)$ at each point $p \in M_I$, and the values of the vector function lie in $T_p[S(p)]$. The gradient of a vector function, which is constant on the orbit, on the other hand, is orthogonal to the orbit, so it must belong to $L(p)$.

Since we have $L(p) = 0$ for an orbit on the Cartan subgroup H_c, the function has an extremum on an orbit constructed on the maximum Abelian subgroup of the group G. The quantities calculated on space-time lattices are functions determined on conjugate classes, i.e., orbits of the group G.

Let us consider, for example, the simplest, single-placket model; as the group G we adopt SU(2).

The thermodynamic potential in this model is

$$Z(\beta) = \int \exp[\beta \operatorname{Re} tr(U_{IJ} U_{JK} U_{KL} U_{LI})] \prod_{b=1}^{4} dU^b,$$

the product of the matrices is taken over a unit square; the index "b" refers to a side of the square; dU^b is an invariant measure on the group SU(2); and $\beta = 1/g^2$, where g is the interaction constant of the gauge fields.

Noting that for any element $g_0 \in G$ there exists an element $g_1 \in G$ such that $g_1 g_0 g_1^{-1} \in H_c$ (Ref. 3), and the invariant integration on the group G reduces to an invariant integration on the subgroup H and the factor-space G/H (Ref. 5), we can rewrite $Z(\beta)$ as

$$Z(\beta) = \int_{SU(2)/U(1)} \prod_{b=1}^{4} d^b t\, \phi^b(t) \int_{U(1)} \exp[\beta \operatorname{Re} tr(\tilde{U}_{IJ}\tilde{U}_{JK}\tilde{U}_{KL}\tilde{U}_{LI})]\, d\tilde{U}^b.$$

The inner integral is over the Cartan subgroup $H_c = \mathrm{U}(1)$ of the group SU(2), while the outer integral is over the factor space SU(2)/U(1), which is isomorphic to S^2. The function ϕ performs the mapping $S^2 \to S^3$. This is a singular function, corresponding to a singular choice of the gauge, in which collective dynamic variables can be singled out in the gauge system, and the long-range dynamics of quantum chromodynamics can be described.[1,2] Abelian dominance arises on the space-time lattice in a singular gauge; for the SU(3) group, for example, the gauge should be chosen in accordance with the

manifold mapping

$$M(SU(3)/U(1) \otimes U(1)) \xrightarrow{\phi} M(SU(3)).$$

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