Collapse of acoustic waves in media with positive dispersion

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It is demonstrated that collapse of acoustic wave occurs in media with positive dispersion. A qualitative description of wave collapse, based on an analysis of the integrals of motion, is proposed.

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The idea of wave collapse is currently widely used in plasma physics and nonlinear optics. The first investigation of this type of phenomenon was the self-focusing of light in a nonlinear dielectric.[1,2] Later, an entire series of analogous phenomena was discovered in plasma physics.[3–6]

A general property of all these systems is the Hamiltonian nature of the equations. This property is the basis for studying wave collapse from a unified point of view. We shall show that the possibility of collapse itself as well as its dynamics are determined by quite rough characteristics of the system: the transformation properties of the Hamiltonian under scale transformations. We shall demonstrate this approach for a new example of collapse of acoustic waves in media with positive dispersion (magnetosonic waves in plasma, phonons in liquid helium, etc.). This problem is especially important for the investigation of collisionless shock waves in magnetoactive plasma, their structure, stability, etc.[7]

We shall restrict our analysis to weakly nonlinear waves when the Kadomtsev-Petviashvili (KP) equation[8] is valid. We shall write this equation in Hamiltonian form

$$u_t = \frac{\partial}{\partial x}\frac{\delta \mathcal{H}}{\delta u}\ , \qquad (1)$$

where the Hamiltonian

$$\mathcal{H} = \int \left(\frac{u_x^2}{2} + \frac{(\nabla_\perp W)^2}{2} - u^3\right) d\mathbf{r}\ ; \quad W_x = u\ .$$

In the one- and two-dimensional cases ($d = 1, 2$), is it integrated by the method of the inverse scattering problem and has an infinite number of integrals of motion.[9] In the three-dimensional case this equation conserves, in addition to $\mathscr{H}$, the momentum $\mathbf{P} = \int u \nabla W d\mathbf{r}$ and the longitudinal projection of the angular momentum M_x.

The analogy to other related phenomena, for example, Langmuir collapse, is already evident at the linear stage of the instability of the one-dimensional soliton of Eq. (1)

$$u(x, t) = \frac{2\nu^2}{\mathrm{ch}^2 \nu (x - 4\nu^2 t)}$$

The increment $\Gamma(k_\perp)$,[10]

$$\Gamma(k_\perp) = \frac{4k_\perp}{\sqrt{3}}\left(\nu^2 - \frac{4k_\perp}{\sqrt{3}}\right)^{1/2},$$

just as for Langmuir waves with $k = 0$, is positive in the finite region $k_\perp < k^* = (\sqrt{3}/4)v^2$. For $k_\perp = k^*$, the instability due to nonlinear effects is stabilized by dispersion terms.

The basic reason for the instability is that the velocity of the soliton decreases with increasing amplitude. For this reason, when the soliton is weakly modulated along the transverse coordinate, the low-amplitude will overtake the high-amplitude sections. As a result, a self-focusing type instability arises. Two-dimensional solitons are similarly unstable relative to bending along the third coordinate.[11]

We shall present simple dimensional considerations of the nonlinear dynamics of the system, based on an analysis of the integrals of motion. We shall restrict the analysis to axially symmetrical distributions, for which $\mathbf{P}_\perp = M_x = 0$.

Let us examine the scale transformations conserving P_x,

$$u(x, \mathbf{r}_\perp) = \alpha^{-1/2} \beta^{(1-d)/2} u(x/\alpha, \mathbf{r}_\perp/\beta)$$

for which the Hamiltonian becomes a function of the parameters α, β:

$$\mathscr{H} = \alpha^{-2} I_1 + \alpha^2 \beta^{-2} I_2 - \alpha^{-1/2} \beta^{(1-d)/2} I_3,$$

$$I_1 = \int \frac{u_x^2}{2} d\mathbf{r}, \quad I_2 = \int \frac{(\nabla_\perp W)^2}{2} d\mathbf{r}, \quad I_3 = \int u^3 \, d\mathbf{r}.$$

The behavior of $\mathscr{H}$ differs depending on the dimensionality of the space. For $d = 2$, $\mathscr{H}(\alpha, \beta)$ has a lower bound: $\mathscr{H}_{\min} = -vP_x/6$ corresponds to a two-dimensional soliton,[11] moving with velocity v. For $d = 3$, the three-dimensional soliton corresponds to a saddle point, so that it is unstable. The Hamiltonian for this solution, as for the three-dimensional Langmuir soliton, is positive $\mathscr{H}_s = vP_x/2$. We emphasize that $\mathscr{H}(\alpha, \beta)$ for $d = 3$ has no lower bound. The unbounded nature of the Hamiltonian is due to nonlinear terms whose role with decreasing scales α, β becomes predominant. To verify this, it is enough to examine the contours of steepest descent of the function $\mathscr{H}(\alpha, \beta)\, \alpha^2/\beta = \text{const}$. Such behavior of $\mathscr{H}(\alpha, \beta)$ also indicates that the general solution must have a self-similar asymptotic form $r_\perp/x^2 = \text{const}$. This situation, viz., the unboundedness of $\mathscr{H}$ with other integrals fixed (number of waves, momentum, etc.), is common for all systems in which the collapse phenomenon exists. The unboundedness fof $\mathscr{H}$ arises due to nonlinear terms that increase with decreasing scales more rapidly than the dispersion terms. As for mechanical systems with friction, the energy principle can be used to describe the evolution of such systems. From this point of view, the collapse process is analogous to the falling of a particle in an infinite potential. Here radiation out of the region with high field concentration (caverns) plays the role of friction. The radiation, on the one hand, should not change greatly the number of quanta, since in the opposite case, due to the decrease in intensity, the reason for the "fall" itself disappears. On the other hand, the radiation must carry away the positive part of the Hamiltonian, due to such a "fall." As a result, singularities, i.e., collapse, appear at separate points. In the presence of collapse, the radiation assumes increasingly shorter wavelengths with the characteristic value of k proportional to the inverse size of the cavern. Thus waves emitted by the caverns form a weakly turbulent background (for them, the dispersion terms exceed the nonlinear terms, $\mathscr{H} > 0$).

Nonradiative collapse can occur in systems in which the nonlinear and dispersion terms in $\mathscr{H}$ transform in the same way under scale transformations. This situation occurs for two-dimensional self-focusing of light due to the Kerr nonlinearity and for supersonic Langmuir and lower hydbrid collapses.

To study the dynamics of the collapse of acoustic waves in media with positive dispersion, we solved the KP equation numerically in an axially symmetrical case. In this case there are a number of difficulties which are related to the nonlocal law $kx(\omega + k_x^3) = -k_\perp^2$ and which lead to large values of the group velocities of the low-order harmonics. For this reason, we developed a new difference scheme with a high degree of accuracy $O(\Delta x^4, \Delta r_\perp^2, \Delta t^2)$, using iterative separation. The efficiency of the

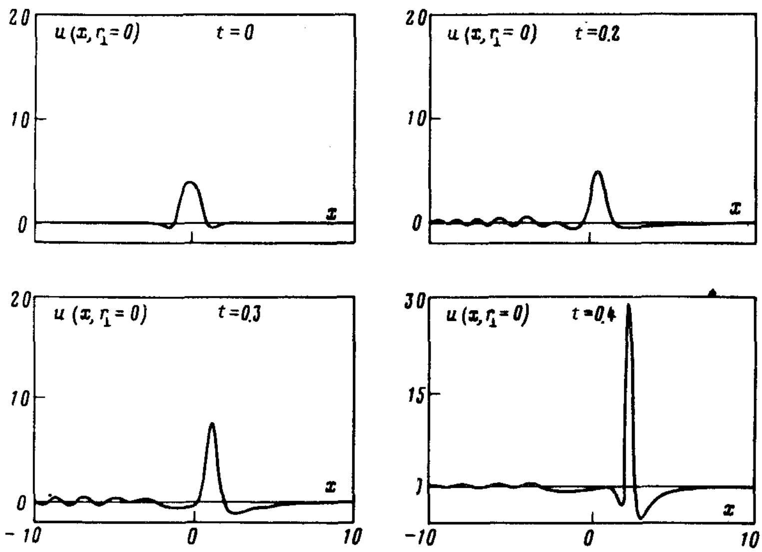

FIG. 1. The distribution of u along the axis $r_\perp = 0$ at successive times.

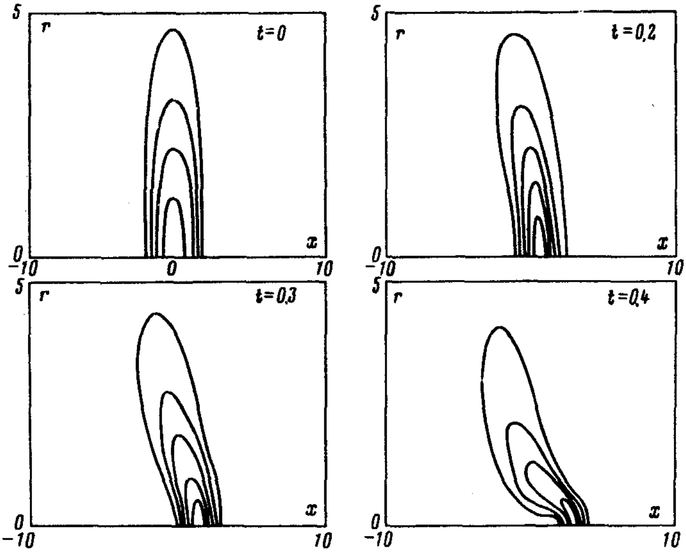

FIG. 2. Contours of the function $u(x, r_\perp)$ at the same times.

scheme was tested for a two-dimensional soliton.[9] To account for possible effects of radiation out of the system, dissipative boundary conditions, which do not conserve $\mathscr{H}$ and P_x, were chosen. In order to monitor the calculation, the fluxes of $\mathscr{H}$ and P_x at the boundaries were calculated and their conservation was a checked. In all variants, the accuracy of conservation of $\mathscr{H}$ was no worse than 5%, while that of P_x was an order of magnitude better.

In the numerical experiments, for values of $\mathscr{H}$ less than the critical value (in this case, it was positive), we observed the formation of caverns, where the amplitude increased by an order of magnitude, i.e., the intensity increased by two orders of magnitude (see Fig. 1). In this case, radiation out of the caverns was observed, the Hamiltonian decreased from 30 to -80, while P_x varied insignificantly ($\sim 10\%$). As with the instability of the leading edge of a soliton, with collapse the center lagged behind the periphery and a U-shaped profile formed (Fig. 2). For $\mathscr{H}$ greater than the critical value, the initial distribution spread out.

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