Proton and neutron magnetic moments in quantum chromodynamics

B. L. Ioffe and A. V. Smilga
Institute of Experimental and Theoretical Physics, Academy of Sciences of the USSR

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The magnetic moments of the nucleons are calculated in nonperturbative quantum chromodynamics without the use of any models. The magnetic susceptibility of the quark condensate plays an important role in determining the magnetic moments. The calculated results agree well with experiment.

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To calculate the magnetic moments of nucleons is one of the foremost problems of quantum chromodynamics and a problem which has yet to be solved. In the present letter we show that these moments can be calculated by generalizing the quantum-chromodynamics sum rules which were proposed in Ref. 1 and extended to baryons in Refs. 2 and 3.

Let us examine the polarization operator

$$\Pi(p) = i\int d^4x e^{ipx} < 0 | T\{ \eta(x), \bar{\eta}(0) \} | 0 >, \quad (1)$$

where

$$\eta_p(x) = (u^a C\gamma_\mu u^b)\gamma_5\gamma_\mu d^c \epsilon^{abc}, \qquad \eta_n(x) = (d^a C\gamma_\mu d^b)\gamma_5\gamma_\mu u^c \epsilon^{abc} \quad (2)$$

are the quark currents with the quantum numbers of the proton and the neutron, and u^a and d^a are the fields of the u and d quarks. To find the magnetic moments of the nucleons we assume that the quarks are in a static electromagnetic field $F_{\mu\nu}$, and we examine the terms in (1) which are linear in $F_{\mu\nu}$.

Proceeding in the spirit of the quantum-chromodynamics sum rules, we will calculate $\Pi(p)$ in the region $p^2 < 0$, $|p^2| \sim 1$ GeV^2, by means of an operator expansion, writing (1) as a series in $1/|p^2|$ with coefficients expressed in terms of the vacuum expectation values of the various operators; in addition, we will write an expression for the same quantity, $\Pi(p)$ in the form of matrix elements between physical states. Two important and distinctive features which arise when we follow this procedure for the polarization operator in the external field distinguish this case from that of a polarization operator without an external field, $\Pi^{(0)}(p)$, which is used to calculate masses.

1) New vacuum expectation values, not present in $\Pi^{(0)}(p)$, arise in the operator expansion for $\pi(p)$ in an external field. Of these expectation values, the most important in our case is $\langle 0|\bar{\psi}_q\sigma_{\mu\nu}\psi_q|0\rangle$, where $\psi_q = ud$. In the presence of an external electromagnetic field we have $\langle 0|\bar{\psi}_q\sigma_{\mu\nu}\psi_q|0\rangle \neq 0$, and this quantity is proportional to $F_{\mu\nu}$ in the linear approximation in the field: $\langle 0|\bar{\psi}_q\sigma_{\mu\nu}\psi_q|0\rangle = \chi_q F_{\mu\nu}\langle 0|\bar{\psi}\psi|0\rangle$ The coefficient χ_q represents the magnetic susceptibility of the quark condensate. We assume that χ_q is proportional to the quark charge: $\chi_q = e_q\chi$. This assumption corresponds to ignor-

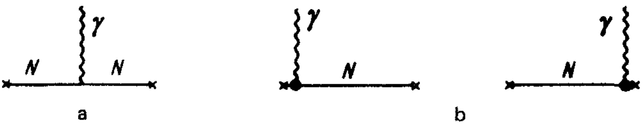

FIG. 1

ing the closed loops,as is done for the vacuum fluctuations of the instanton type and confirmed by lattice calculations. In writing the operator expansion we should bear in mind that since we are interested in terms which are linear in the field the characteristic quantity which determines the power of $1/p^2$ for the vacuum expectation values of the operators proportional to $F_{\mu\nu}$ is not the ordinary dimensionality d but the quantity $d_{\rm eff} = d - 2$. For $\langle 0|\bar{\psi}_q \sigma_{\mu\nu} \psi_q|0\rangle$ we have $d_{\rm eff} = 1$; i.e., the dimensionality is minimal in this case. The next highest dimensionality is $d_{\rm eff} = 3$, since the quark condensate $\langle 0|\bar{\psi}\psi|0\rangle$ also has a vacuum expectation value, $(g_s/2) < 0|\bar{\psi}_q G^a_{\mu\nu} \lambda^a \psi_q|0\rangle = e_q \kappa F_{\mu\nu} \langle 0|\bar{\psi}\psi|0\rangle$, where $G^a_{\mu\nu}$ is the gluon field. A large number of operators with unknown vacuum expectation values have the dimensionality $d_{\rm eff} = 5$.

2) In writing $\Pi(p)$ in terms of the matrix elements between physical states we are primarily interested in the term $\langle 0|\eta|N\rangle\langle N|j^{\rm el}|N\rangle\langle N|\bar{\eta}|0\rangle(p^2 - m^2)^{-2}$, which corresponds to the diagram in Fig. 1a. This term is expressed in terms of the magnetic moment of the nucleon and has a second-order pole in $p^2 - m^2$ (m is the mass of a nucleon). In addition, however, there are terms of first order in $p^2 - m^2$, which correspond to the diagrams in Fig. 1b, where the contribution of the intermediate single-nucleon state has been eliminated at the heavy vertex. A Borel transformation of the terms of the first type gives rise to a factor $(1/M^2)\exp(-m^2/M^2)$, while one for terms of the second type gives rise to $\exp(-m^2/M^2)$. The terms of the second type are thus not suppressed exponentially with respect to those of the first type and definitely must be taken into account.[1)]

The terms in $\Pi(p)$ in (1) which are linear in $F_{\mu\nu}$ are expressed in terms of three tensor structures: $\sigma_{\mu\nu}\hat{p} + \hat{p}\sigma_{\mu\nu}, \sigma_{\mu\nu}$ and $(p_\mu \gamma_\nu - p_\nu \gamma_\mu)\hat{p}$. We will consider the sum rules for only the first and third structure; the second structure is inconvenient here because the leading term turns out to be suppressed in the procedure used below, and everything is determined by the unknown vacuum expectation values of higher dimensionalities. The third structure contains two more momenta in the numerator than are present in the second structure; this circumstance makes the vacuum expectation values of higher dimensionalities numerically less important and weakens the role played by the high-lying states (the continuum) in the sum rules. After the Borel transformation the sum rules become (for the proton)

$$e_u M^4 E_1(M) L^{4/9} + \frac{a^2 L^{4/3}}{3M^2}\left[-\left(e_d + \frac{2}{3}e_u\right) + \frac{e_u}{3}\kappa + 2e_u \chi \left(M^2 - \frac{m_0^2}{8}\right) L^{-16/27}\right]$$

$$= (\tilde{\lambda}^2/4)\exp(-m^2/M^2)[(\mu_p/M^2) + A_p] \qquad (3)$$

$$ma\,\{(e_u + \frac{1}{2}e_d)L^{8/9} - \frac{1}{3}e_d\,\chi M^2\,[E_0(M) + \frac{7\pi^2}{36M^4}\langle 0|\frac{\alpha_s}{\pi}G^2_{\mu\nu}|0\rangle]\,L^{8/27}\}$$

$$= (\tilde{\lambda}^2/4)\exp(-m^2/M^2)[(\mu_p^a/M^2) + B_p] \qquad (4)$$

where

$$a = (2\pi)^2|\langle 0|\bar{\psi}\psi|0\rangle|,\; g_s\langle 0|\bar{\psi}\sigma_{\mu\nu}(\lambda^a/2)G^a_{\mu\nu}\psi|0\rangle = m_0^2\langle 0|\bar{\psi}\psi|0\rangle,$$

$$E_0(M) = 1 - \exp(-W^2/M^2),\; E_1(M) = 1 - \exp(-W^2/M^2)(1 + W^2/M^2), \qquad (5)$$

$$L = \ln(M/\Lambda)/\ln(\mu/\Lambda) \qquad \Lambda \approx 100\ \text{MeV},\; \tilde{\lambda}^2 = 2(2\pi)^4\lambda^2,$$

λ is the amplitude for the proton transition to the current η_p, defined in Refs. 2 and 3, W is the continuum threshold, $\mu \approx 0.5$ GeV is the normalization point, and μ_p and μ_p^a are the total and anomalous magnetic moments of the proton. The constants A_p and B_p on the right sides of (3) and (4) reflect the diagrams in Fig. 1b. The sum rules for the neutron can be found from (3) and (4) through the replacement $e_u \rightleftarrows e_d; \mu_p, \mu_p^a \to \mu_n; A_p \to A_n; B_p \to B_n$.

To get rid of the unknown constants A and B, we apply the operator $\partial/\partial(1/M^2) + m^2$ to (3) and (4). We then multiply Eq. (3) in the proton version by e_d and in the neutron version by e_u and take the difference. We multiply (4) in the proton version by e_u and in the neutron version by e_d and again take the difference. As a result, we find two equations relating μ_p and μ_n which do not contain the unknown parameters χ and κ:

$$\mu_p e_d \quad \mu_n e_u = \frac{4}{3}a^2\tilde{\lambda}^{-2}\exp(m^2/M^2)(e_u^2 - e_d^2)[\partial/\partial(1/M^2) + m^2]\frac{1}{M^2}L^{4/3}$$

$$\mu_p^a e_u - \mu_n e_d = 4a\,m\,\tilde{\lambda}^{-2}\exp(m^2/M^2)(e_u^2 - e_d^2)[\partial/\partial(1/M^2) + m^2]L^{8/9}. \qquad (6)$$

We ignore the anomalous dimensionalities and the continuum; then we find[2] $\tilde{\lambda}^2\exp(-m^2/M^2) = 2aM^4/m$. Setting $M = m$ in (6), and solving the linear equations for μ_p and μ_n, we find ($e_u = 2/3, e_d = -1/3$)

$$\mu_p = \frac{8}{3}(1 + \frac{1}{6}\frac{a}{m^3}),\; \mu_n = -\frac{4}{3}(1 + \frac{2}{3}\frac{a}{m^3}). \qquad (7)$$

The magnitude of the quark condensate, a, is known: $a = 0.55$ GeV^3. Substitution of the numerical values into (7) yields $\mu_p = 2.96$ and $\mu_n = -1.93$, which are to be compared with the experimental values $\mu_p = 2.79$ and $\mu_n = -1.91$. Incorporating the anomalous dimensionalities and the continuum does not alter the results within the presumed error ($\sim 10\%$). From (3) and (4) we can determine the magnetic susceptibility of the quark condensate, $\chi = (300 \pm 50\ \text{MeV})^{-2}$. Here χ is numerically large, so that the magnetic susceptibility of the quark condensate must be taken into account in examining the static electromagnetic properties of hadrons in quantum chromodynamics. We can also find the constants A and B on the right sides of (3) and (4): $A_p = 5.3$ GeV^{-2}, $A_n = -2.4$ GeV^{-2}, $B_p = 2.2$ GeV^{-2}, and $B_n = -3.6$ GeV^2. With $M^2 \sim 1$ GeV^2, their contributions to (3) and (4) are comparable to those of the terms $\mu_{p,n}/M^2$.

[1)]Terms of the same type arise in lattice calculations of the magnetic moments, and for this reason lattice calculations which ignore these terms[4,5] appear to us to be incorrect.

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