Phase transitions in the Euclidean and Hamiltonian approaches to lattice gauge theories at a finite temperature

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The thermodynamic properties of a gluon gas at finite temperatures are studied in a gauge field theory on a lattice with SU(2) symmetry. The Monte Carlo method has been used to study the temperature dependence of a Wilson string on symmetric ($a_t = a_s$) and asymmetric ($a_t \neq a_s$) lattices. The relationship between the interaction constants in the two approaches is found, as is the ratio of renormalization constants.

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Significant progress has recently been made in research on the quark-gluon state of matter in physics of phase transitions of the confinement-deconfinement type. To a significant extent this progress can be attributed to the formulation of the lattice gauge field theory.[1]

The use of a lattice regularization in the derivation of gauge field theories has made it possible to employ the Monte Carlo method for a numerical study of the interaction of quarks and gluons and of the various phases of the state of matter.

Various models have been proposed for the lattice gauge field theories.

The physical quantities determined in the various renormalization schemes can be compared only if the relationship between the constants in the different models is known.

In this letter we report a direct, nonperturbative comparison of the bare interaction constants by the Monte Carlo method in the Euclidean[1] and Hamiltonian[2,4] approaches at a finite temperature.

The partition functions in the weak-coupling limit have recently been calculated in these two approaches by a background-field method.[3–5] These calculations, which were carried out in the single-loop approximation (in the limit $a \to 0$, where a is the lattice step), revealed the relationship between the interaction constants. This calculation method is applicable only at small values of $g^2(a)$, while the most interesting events (phase transitions at finite temperatures, etc.) occur at $g^2(a) \sim 2$. It is therefore worthwhile to carry out a direct, nonperturbative calculation of the relationship between the interaction constants. These calculations, in addition to constituting a test of the theory for self-consistency, can reveal the relationship between the interaction constants outside the weak-coupling region. We will restrict the discussion to SU(2) symmetry with periodic boundary conditions.

In the gluon sector of the Euclidean lattice field theory the action is[1]

$$S_E(U) = \beta_E \sum_P \left(1 - \frac{1}{2} \operatorname{Sp} U_p\right), \qquad U_P = U_{ij} U_{jk} U_{kl} U_{li}. \tag{1}$$

Here U_{ij} is an element of the SU(2) group corresponding to the edge (i, j), and $\beta_E(a) = 4/g_E^2(a)$, where $g_E(a)$ is the bare interaction constant, which depends on the distance (a) between two adjacent nodes i and j. The summation in (1) is over all the elementary squares (placquets). In the Euclidean approach the distance a does not depend on the direction of the edge (i, j). The partition function Z is defined by the integral

$$Z = \int [dU] e^{-S_E(U)}, \tag{2}$$

where

$$[dU] = \prod_{(i,j)} dU_{ij}$$

and dU_{ij} is the Ter Haar measure on the SU(2) group. The expectation value of any quantity $O(U)$ is

$$\langle O \rangle = Z^{-1} \int [dU] O(U) e^{-S_E(U)}. \tag{3}$$

In the Hamiltonian approach we work from the Hamiltonian[2,4]

$$\hat{H} = \sqrt{\frac{g_t^2}{g_s^2}} \frac{g_H^2}{2a} \left\{ \sum_{edges} \mathbf{E}^2 + \frac{2}{g_H^4} \sum_{P_s} \operatorname{Sp}(2 - \hat{U}_{P_s} - \hat{U}_{P_s}^+), \right. \tag{4}$$

where $g_H^2 = g_t g_s$ and the operators $\mathbf{E}(\mathrm{x};\mathrm{y})$ and $\hat{U}_{P_s} = \prod_{\text{placquet}} \hat{U}_{ij}$ are determined by the commutation relations

$$\begin{aligned} [E^\alpha(\mathrm{x};\mathrm{x}+a\mathrm{e}_k), E^\beta(\mathrm{y};\mathrm{y}+a\mathrm{e}_{k'})] &= i\epsilon^{\alpha\beta\gamma} E^\gamma(\mathrm{x};\mathrm{x}+a\mathrm{e}_k)\delta_{\mathrm{x};\mathrm{y}}\delta_{\mathrm{e}_k,\mathrm{e}_{k'}}, \\ [E^\alpha(\mathrm{x};\mathrm{x}+a\mathrm{e}_k), \hat{U}(\mathrm{y};\mathrm{y}+a\mathrm{e}_{k'})] &= \frac{1}{2}\sigma^\alpha \hat{U}(\mathrm{y};\mathrm{y}+a\mathrm{e}_{k'})\delta_{\mathrm{x};\mathrm{y}}\delta_{\mathrm{e}_k,\mathrm{e}_{k'}}, \end{aligned} \tag{5}$$

The sum in (4) is over coordinate space. The expectation value of any operator $\hat{O}$ is defined by

$$<\hat{O}> = Z^{-1}\,\mathrm{Sp}\left(\hat{O}e^{-\frac{1}{\theta}\hat{H}}\right);\qquad Z = \mathrm{Sp}\,e^{-\frac{1}{\theta}\hat{H}} \tag{6}$$

The expression for the partition function in (6) can be written by analogy with (2):

$$Z = \int [dU]\, e^{-S_H(U)}, \tag{7}$$

where

$$S_H(U) = \beta_H\left\{\bar{\xi}^{-1}\sum_{P_s}\left(1-\frac{1}{2}\mathrm{Sp}U_{P_s}\right)+\bar{\xi}\sum_{P_t}\left(1-\frac{1}{2}\mathrm{Sp}U_{P_t}\right)\right\}. \tag{8}$$

In (8), we have $\beta_H = 4/g_H^2$, $\xi = a_s/a_t$ and $\bar{\xi} = \sqrt{(g_s^2/g_t^2)}\xi \equiv \eta\xi$ and expression (7) is a definition of the partition function in the limit $\xi \to \infty$ ($a_t \to 0$ with a_s fixed). The temperature $\Theta = 1/a_t N_t$ must be fixed. In the limit $a_s \to 0$ the bare constant $g_{E/H}(a)$ behaves in the established manner (see Ref. 5, for example):

$$g^2_{E/H}(a) \underset{a\to 0}{=} \left[b_0 \ln\frac{1}{a^2\Lambda^2_{E/H}} + \frac{b_1}{b_0}\ln\ln\frac{1}{a^2\Lambda^2_{E/H}} + O(g^2_{E/H})\right]^{-1}. \tag{9}$$

For SU(2) symmetry we have $b_0 = (11/3)(1/8\pi^2)$ and $b_1 = (34/3)(1/8\pi^2)^2$.

As the object of the calculations, we adopt the temperature Wilson loop

$$L = \frac{1}{2}\,\mathrm{Sp}\Pi\, U_{ij} \equiv \frac{1}{2}\mathrm{Sp}\,U_{i_1 i_2}U_{i_2 i_3}\cdot\ldots\cdot U_{i_{N_t-1}i_{N_t}}, \tag{10}$$

where the notes $i_1,\ldots,i_N$, lie on a common line along the time axis. The expectation value of the temperature Wilson loop is the order parameter for global $Z(2)$ symmetry and is convenient for describing phase transitions.[6,7]

Physically, the expectation value of L is the partition function (Z_q) of a system of a Yang-Mills gas with a source at rest:

$$< L > \equiv Z_q = e^{-\frac{1}{\theta}F_q}, \tag{11}$$

where F_q is the free energy in the presence of a source at rest (a quark).[8-10] We calculated the expectation value of L in the Euclidean approach on a 3×7^3 lattice and in the Hamiltonian approach on a 15×7^3 lattice with $\xi = 5$ and various values of η. Figure 1 shows the behavior of the Wilson loop in the Euclidean approach (the solid curve) and in the Kogut-Susskind Hamiltonian approach with $\eta = 1$ (the dashed curve). The curve derived in the Kogut-Susskind Hamiltonian approach lies above that derived in the Euclidean approach, indicating that we must take into account the deviation from unity of the second Hamiltonian variable η. In either case the depen-

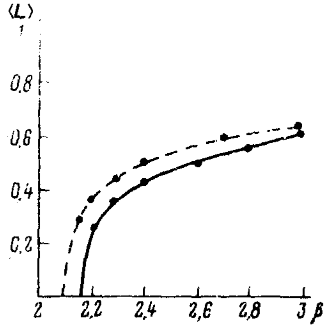

FIG. 1.

dence $\langle L \rangle$ on β (β_E and β_H, respectively) is described well by

$$\langle L \rangle = \frac{\tau^{\alpha}}{\tau^{\alpha} + \gamma}, \tag{12}$$

where $\tau \equiv (\beta - \beta_c)/\beta_c$. The values of α and γ agree, within the errors, and are given by

$$\alpha \simeq 0{,}5; \qquad \gamma \simeq 0{,}4. \tag{13}$$

The values of β_{cE} and β_{cH}, on the other hand, are different:

$$\beta_{cE} \simeq 2{,}15; \quad \beta_{cH} \simeq 2{,}1. \tag{14}$$

Using expressions (12)–(14) we can easily show that the values of β_E and β_H are related by the simple relation

$$\beta_H - \beta_E = \Delta\beta = \beta_{cH} - \beta_{cE}\,. \tag{15}$$

To study the role played by the quantum corrections to the Hamiltonian, we examined the η dependence of the Wilson loop. Figure 2 shows $\langle L \rangle$ as a function of η for $\beta = 2.4$ on a 15×7^3 lattice with $\xi = 5$. The calculations carried out in the single-loop approximation in Refs. 4 and 5 predict the value $\eta \simeq 0.88$. Using this value of η, we

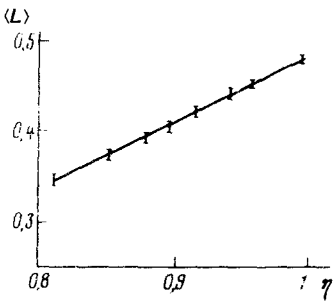

FIG. 2.

can calculate the ratio Λ_H/Λ_E; working from (9) we find $\Lambda_H/\Lambda_E \simeq 0.87$, in good agreement with Refs. 4 and 5.

Our calculations thus show that in the region of intermediate and weak coupling, β_E, $\beta_H \sim 2$, on a lattice with an asymmetric step, we must take into account the quantum corrections, which lead to a deviation of the second Hamiltonian constant $\eta = \sqrt{g_s^2/g_t^2}$ from unity. This deviation reaches 10–20%. In this case the dependence of β_E and β_H on the lattice step agrees with the asymptotic-freedom expressions, and the resulting ratio of renormalization constants Λ_H/Λ_E agrees surprisingly well with the value calculated for Λ_H/Λ_E in the single-loop approximation in the weak-coupling case in Refs. 4 and 5.

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