# Dynamical chaos and spontaneous symmetry breaking in anharmonic systems excited by a periodic external force

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It is shown that in anharmonic systems having a symmetric potential under excitation by a periodic force, a necessary condition for dynamical chaos to arise through period doubling is the spontaneous breaking of symmetry. The nature of the low-frequency peak, which appears in the Fourier spectrum at the onset of dynamical chaos, is discussed.

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The concept of dynamical chaos—the chaotic behavior of finite-dimensional dynamical systems—is penetrating more and more widely into various areas of physics. Particular interest attaches to the recently discovered universality of the period-doubling mechanism for the onset of dynamical chaos.[1] Many papers[2–4] have been devoted to the study of this mechanism in systems whose behavior is described by the equation

$$\ddot{\eta} + \gamma\dot{\eta} + \frac{\partial U}{\partial \eta} = B\cos\omega t\ , \tag{1}$$

where $U = \frac{1}{2}a\eta^2 + \frac{1}{4}b\eta^4$ is the potential, $\gamma$ is the damping coefficient, $B$ and $\omega$ are the amplitude and frequency, respectively, of the external force, and $a$ and $b$ are real numbers. Equation (1) describes the behavior of an anharmonic oscillator under the action of a periodic force in cases A ($a\geqslant 0,\ b>0$; Fig. 1a) and B ($a>0,\ b<0$; Fig. 1b),

  

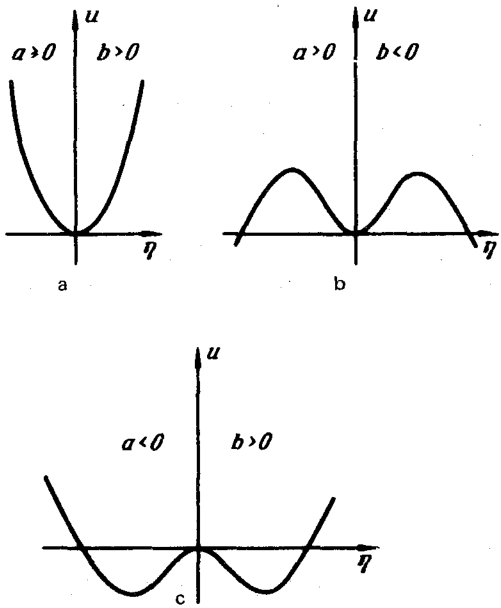


FIG. 1.

and the dynamics of the order parameter near a second-order phase transition in the presence of a single soft mode in case c ($a<0,b>0$; Fig. 1c).

Our goal in this study is to show that the onset of dynamical chaos in systems described by Eq. (1) for all three cases A–C conforms to a common principle: the doubling of the period and the dynamical chaos arise when the symmetry is spontaneously broken. Let us define what we mean by spontaneous symmetry breaking in anharmonic systems excited by a periodic force. To do this we introduce the quantity

$$<\eta> = \lim_{t\to\infty} \frac{1}{t} \int_0^t \eta(\xi)\, d\xi, \tag{2}$$

which for periodic solutions is the constant term in the Fourier expansion of the solution:

$$\eta\ (t) = <\eta> + \frac{1}{2} \sum_k [\,\eta_k \exp(i\phi k t) + \eta_k^* \exp(-i\phi k t)\,], \tag{3}$$

where $T=2\pi/\phi$ is the period of the solution $\eta(t)$, $\phi=(m/n)\omega$, and $m$ and $n$ are integers. By studying the periodic solution of Eq. (1) in the phase plane $\eta,\dot\eta$, one is convinced that for $\langle\eta\rangle=0$ the solutions [such as Eq. (1) itself] are invariant with respect to the symmetry $S$: $\eta'=-\eta, \dot\eta'=-\dot\eta, t'=t+\pi$, while for $\langle\eta\rangle\neq 0$ this is no longer the case. We therefore call the solutions with $\langle\eta\rangle=0$ symmetric and those with $\langle\eta\rangle\neq 0$ asymmetric, and we call $\langle\eta\rangle$ the order parameter. We understand the term spontaneous symmetry breaking to mean a transition due to loss of stability from the symmetric solution to the stable asymmetric solution (or vice versa) upon a change in the parameter $B$ or $\omega$.

An important property of spontaneous symmetry breaking at the transition from the symmetric to the asymmetric solution is the growth in the amplitude of the even harmonics from zero to a finite value. In a study of case A, Novak and Frehlich[4] noticed that the appearance of the even harmonics presage the period-doubling–dynamical-chaos sequence and, moreover, stressed the universality of this phenomenon, but they did not associate it with spontaneous symmetry breaking. We shall show that the appearance of the even harmonics is in fact due to the transition to the asymmetric solution. We substitute (3) into (1) and obtain an equation for $\langle\eta\rangle$:

$$a\langle\eta\rangle + b\langle\eta\rangle^3 + \frac{3}{2}b\langle\eta\rangle\sum_k|\eta_k|^2 + \frac{3}{8}b\sum_{k,l}(\eta_k\eta_l\eta^*_{k+l} + \eta^*_k\eta^*_l\eta_{k+l}) = 0. \tag{4}$$

For simplicity, let us consider only periodic solutions with the period of the external force. Then each term in the last expression in (4) contains an even harmonic. If the amplitudes of the even harmonics are equal to zero, then in case A, for example, Eq. (4) has only the solution $\langle\eta\rangle = 0$. The appearance of asymmetric solutions is possible only if the amplitudes of the even harmonics are nonzero. For cases C and B the amplitudes of the even harmonics are also proportional to $\langle\eta\rangle$.

By studying the stability of the symmetric periodic solutions, whose Fourier spectrum contains only odd harmonics and has $\langle\eta\rangle = 0$, one is convinced that there is no instability with respect to a doubling of the period. We shall show that the period-doubling sequence arises in the region of stable asymmetric solutions upon a decrease in $\langle\eta\rangle$. Let us consider case C. This case is unique in that the spontaneous symmetry breaking can be studied even in the approximation of a single harmonic. For cases A and B the spontaneous symmetry breaking is revealed only when several harmonics are taken into account. The equations for $\langle\eta\rangle$ and $\eta_1$ in case C, for $a = -1/2$, $b = 1/2$,and $\omega = 1$ are

$$\langle\eta\rangle(\langle\eta\rangle^2 - 1 + 3|\eta_1|^2) = 0, \tag{5}$$

$$|\eta_1|^2\left[\left(-\frac{3}{2} + \frac{3}{4}\langle\eta\rangle^2 + \frac{3}{8}|\eta_1|^2\right)^2 + \gamma^2\right] = B^2. \tag{6}$$

For $B \ll 1$ Eqs. (5) and (6) have two stable periodic solutions with $\langle\eta\rangle \neq 0$, which are localized in each of the potential wells, and unstable solutions with $\langle\eta\rangle = 0$. As $B$ increases, $\langle\eta\rangle$ goes to zero. At $B \cong 1$ the asymmetric solutions vanish and the symmetric solutions become stable, i.e., spontaneous symmetry breaking occurs. Let us examine the stability of the asymmetric oscillations with period $2\pi$:

$$\delta\ddot{\eta} + \gamma\delta\dot{\eta} + \left[\langle\eta\rangle^2 - 3\langle\eta\rangle\eta_1\cos(t+\beta_1) - \frac{3}{4}|\eta_1|^2\cos(2t+\beta_2)\right]\delta\eta = 0, \tag{7}$$

where $\delta\eta$ is a small deviation from the periodic solution, and $\beta_1$ and $\beta_2$ are the phases of the oscillations. Equation (7) is analogous to the equation for a linear pendulum of natural frequency $\langle\eta\rangle$ under parametric pumping by a force with unit frequency. A decrease in $\langle\eta\rangle$ with an increase in B leads to a parametric instability, i.e., to the

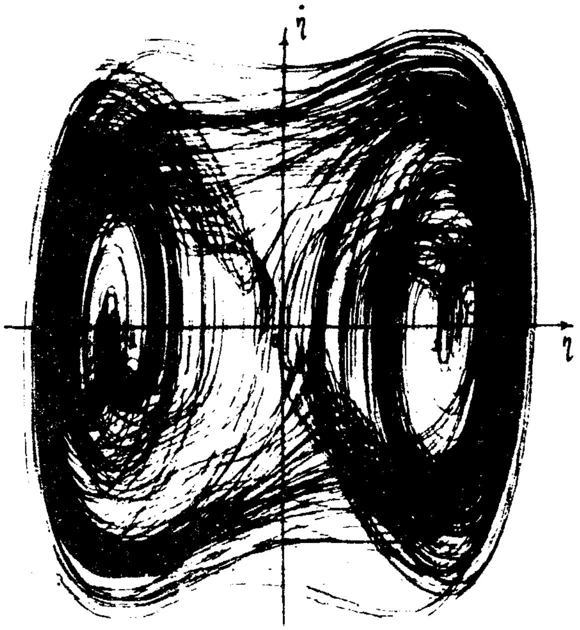

FIG. 2.

growth of fluctuations with frequency 1/2. This in turn leads to the appearance of oscillations with a doubled period. Subsequent doublings of the period are also associated with the parametric instability. Thus one expects dynamical chaos to arise in the immediate region of the spontaneous symmetry breaking. We checked this out with the aid of a computer.

In fact, for $\gamma \lesssim 0.1$ the transition from the asymmetric to the symmetric solutions was accompanied by the onset of dynamical chaos. For example, as $B$ is increased at $\gamma = 0.1$, asymmetric oscillations with periods of 1, 2, 4, 8, and 16 are observed in each of the potential wells, and then at $B = 0.182$ dynamical chaos appears. The solution corresponding to dynamical chaos is symmetric, i.e., according to definition (2) $\langle \eta \rangle = 0$ (Fig. 2), while the amplitudes of the even harmonics in the Fourier spectrum (Fig. 3) have discontinuously fallen almost to zero. The sharp peak at low frequencies is an important characteristic of dynamical chaos. The emergence of this peak is evidently due to random jumps between the asymmetric solutions formed as a result of the spontaneous symmetry breaking.[5] A further increase in the parameter $B$ past the region of dynamical chaos gives rise to symmetric oscillations with a period of 3 or 1.

For $\gamma \lesssim 1$ the spontaneous symmetry breaking is not accompanied by the onset of dynamical chaos.

Conclusions: 1) In anharmonic systems with a symmetric potential under excitation by a periodic force, a necessary condition for the appearance of dynamical chaos is the spontaneous breaking of symmetry, whereby a sequence of period-doubling bifurcations occurs upon the transition from the asymmetric to the symmetric solutions. 2) The onset of dynamical chaos is accompanied by the appearance of a low-

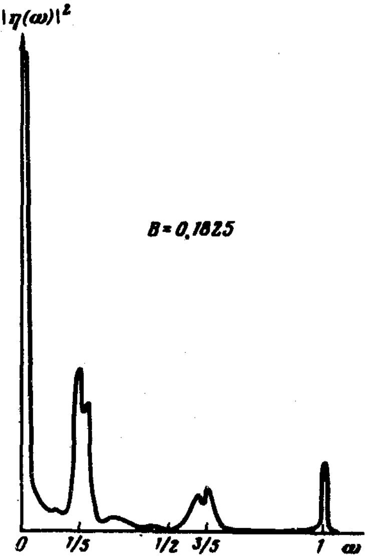


FIG. 3.

frequency peak in the Fourier spectrum. 3) The results of this paper are of interest in dynamical studies of phase transitions.

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