Axial and conformal anomalies in supergravity

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(Submitted 7 April 1983)
Pis'ma Zh. Eksp. Teor. Fiz. **37**, No. 10, 509–512 (20 May 1983)

The axial (triangle) anomaly of antisymmetric tensors is shown to be nonzero. This result is used for a correct determination of the axial anomaly, which turns out to be equal to the conformal anomaly in supergravities with $N = 1,...,8$. At $N \geqslant 3$, both anomalies disappear.

PACS numbers: 04.60. + n, 11.30.Pb, 12.25. + e

INTRODUCTION

1. The absence of anomalies in quantum field theory is an important test in the derivation of a relativistic theory of elementary particles. Anomalies present a problem in supergravity, offered as a candidate for the role of a unified theory of all fundamental interactions: It is believed that the conformal and axial anomalies in a gravitational field do not form a multiplet,[1,2] and the axial anomaly, in contrast with the conformal anomaly,[3,2] does not disappear in supergravities with[2] $N \geqslant 3$. In this letter we show that this problem can be resolved by incorporating an axial anomaly of an antisymmetric tensor ("notoff")[4] and by refining the definition of the axial anomaly in supergravity.

2. Axial anomaly of an antisymmetric tensor. It has previously been believed that the axial anomaly occurs only in the case of fermions, i.e., fields of spin 1/2 and 3/2, while for boson fields, including antisymmetric tensors described by second-order Lagrangians, the axial anomaly does not occur.[5,2] To analyze this point we note that in the classical theory of the field $A_{[\mu\nu]}$ which is interacting with a gravitational field there is a conserved axial current $j^5_\mu = D^\lambda A^*_{\lambda\nu} A_{\mu\nu}$, where $A^*_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\lambda\delta} A_{\lambda\delta}$ (by virtue of the equations of motion and the auxiliary condition $D^\mu A_{\mu\nu} = 0$). We also consider the quantum Lagrangian of the field $A_{\mu\nu}$ (in a harmonic gauge),

$$\mathcal{L}_{qu} = -\sqrt{g}\,(D^\mu A^*_{\mu\nu})^2 - \sqrt{g}(D^\mu A_{\mu\nu})^2 + \mathcal{L}_{gh}\,, \tag{1}$$

where the ghosts are vector and scalar particles. For the dual transformation $A_{\mu\nu} \to A^*_{\mu\nu}$ we have

$$A_{\mu\nu} \to A^*_{\mu\nu}$$

$$\delta\mathcal{L}_{qu} = -2\sqrt{g}\,D^\mu A^*_{\mu\nu} D^\lambda A_{\lambda\nu} \equiv -\sqrt{g}\,D^\mu j^5_{\mu\,qu}(x) + \frac{\delta S_{qu}}{\delta A_{\mu\nu}(x)} A^*_{\mu\nu}(x)\,, \tag{2}$$

where

$$j^5_{\mu\,qu} = D^\lambda A^*_{\lambda\nu} A_{\mu\nu} + D^\lambda A_{\lambda\nu} A^*_{\mu\nu}\,. \tag{3}$$

We now consider the average identity (2), $\langle \delta L(x)\rangle$, here $\langle F(x)\rangle$ is $\int dA_{\mu\nu}\, d\Phi_{gh} \exp\{iS_{qu}\} F(x)$. Using the substitution of variables $\delta A_{\mu\nu} = D_\mu \xi_\nu$ in the corresponding functional integral, we find a Ward identity in the gravitational field:

$$< D^{\mu} A^{*}_{\mu\nu}(x) D^{\lambda} A_{\lambda\delta}(y) > = 0 \cdot \tag{4}$$

The expression $\langle [\delta S_{qu}/\delta A_{\mu\nu}(x)]\, A^{*}_{\mu\nu}(x)\rangle$ is poorly limited. By a regularization procedure we find for it[1,5]

$$< \frac{\delta S_{qu}}{\delta A_{\mu\nu}(x)} A^{*}_{\mu\nu}(x) > \; = \; < A^{+}_{\mu\nu}\, \Delta^{+\,\mu\nu,\,\lambda\delta} A^{+}_{\lambda\delta} - A^{-}_{\mu\nu}\, \Delta^{-\,\mu\nu,\,\lambda\delta} A^{-}_{\lambda\delta} >$$

$$= \sqrt{g}\,(b_4\,[1,0] - b_4[\,0,1]\,) = \frac{\sqrt{g}}{48\,\pi^2} R^{*}_{\mu\nu\lambda\delta}\, R^{\mu\nu\lambda\delta}(x)\,, \tag{5}$$

where $A^{\pm}_{\mu\nu} = A_{\mu\nu} \pm A^{*}_{\mu\nu}$, and $\Delta^{\pm} A^{\pm} = 0$ is the equation in the gravitational field for the self-dual (+) and anti-self-dual (—) parts of the field $A_{\mu\nu}$. It thus follows from (2), (4), (5) that if there is *gauge invariance* with respect to the gravitational field and with respect to the field $A_{\mu\nu}$ [in the form in (4)], then we have a triangle axial anomaly for the latter:

$$<D^{\mu} j^{5}_{\mu\, qu}(x) > = \frac{1}{48\,\pi^2}\; R^{*}_{\mu\nu\lambda\delta}\; R^{\mu\nu\lambda\delta}, \tag{6}$$

where $j^{5}_{\mu qu}$ is defined in (3). Equation (6) is a new (in comparison with Ref. 5) local version of the Rokhlin-Thom-Hirzebruch theorem,[6] which relates the number ($n_2^{\pm}$) of harmonic (anti-)self-dual 2-forms with the signature of the manifold, τ) and the Pontryagin number P:

$$n^{+}_{2} - n^{-}_{2} = \tau = P/3\,,\; P = \frac{1}{16\,\pi^2}\; \int d^4 x \sqrt{g}\, R^{*}_{\mu\nu\lambda\delta}\, R^{\mu\nu\lambda\delta} \tag{7}$$

The relationship between (7) and (6) is the same as that between the Atiyah-Singer theorem[7] regarding the index of the Dirac operator and the fermion γ^5 anomaly.[8]

The field $A_{[\mu\nu\lambda]}$ makes a double negative (with respect to $A_{\mu\nu}$) contribution to the axial anomaly, since ghosts $A'_{[\mu\nu]}$, $A_{[\mu\nu]}$ appear upon its quantization.

3. Axial and conformal anomalies in supergravity. To determine the anomalies we begin with the question of just which field transformations are to be studied and whether the corresponding local substitutions of variables, $\phi^{i'} = \phi^{i} + \delta\phi^{i}(x)$, are permissible in the functional integral. The general equation for the conformal-chiral anomalies of some set of fields in a gravitational field is

$$\int d\phi^{i} \exp i\, \{ S\,[\,\phi^{i},\, g_{\mu\nu}\,]\,\}\; \frac{\delta S}{\delta\phi^{i}(x)}\, \delta\,\phi^{i}(x) = \delta f(x) < T^{\mu}_{\mu}(x) > + \, i\delta g(x) < D_{\mu} j^{5} > . \tag{8}$$

For fields describable by a reducible representation of the Lorentz group $\phi\,[A,B\,]$, the conformal-chiral transformations are $\delta\phi\,[A,B\,] = (\delta f(x) \pm i\delta g(x)\; \phi\,[A,B\,]$ (+ for $A \geqslant B$ and — for $A \leqslant B$). These anomalies were calculated in Ref. 1 for the case $R = R_{\mu\nu} = 0$:

$$(-1)^{2(A+B)} < T^{\mu}_{\mu}(x) > = b_4\,[A,B] + b_4[B,A] \equiv (\alpha_{+} + \alpha_{-})\, \frac{1}{32\pi^2} (C^2_{+} + C^2_{-}),$$

$$(-1)^{2(A+B)} < D^{\mu} j^{5}_{\mu}(x) > = b_4\,[A,B] - b_4\,[B,A] \equiv (\alpha_{+} - \alpha_{-})\, \frac{1}{32\pi^2} (C^2_{+} - C^2_{-}), \tag{9}$$

where $C_\pm$ are the (anti-)self-dual parts of the Weyl tensor, and

$$\alpha_+[A,B]=\frac{4^{A+B}}{90}\left[1-\frac{15}{2}A-\frac{45}{2}A\,(2A-1)\right];$$
$$\alpha_-[A,B]=\frac{4^{A+B}}{90}\left[1-\frac{15}{2}B-\frac{45}{2}B\,(2B-1)\right]. \tag{10}$$

If we were to use these expressions as in Ref. 1—if we were to sum the axial anomalies of the various spinor fields with a unit chiral weight—then we would find (ignoring the antisymmetric tensors)

$$<D^\mu j^5_\mu(x)>=(21N_{3/2}-N_{1/2})\frac{1}{24\cdot 32\pi^2}(C_+^2-C_-^2)\ , \tag{11}$$

where $N_{3/2}$ is the number of gravitinos, and $N_{1/2}$ is the number of fields with $s=1/2$. This anomaly does not disappear in any of the theories presently under consideration. The situation is not improved by incorporating antisymmetric tensors. In a study of supersymmetry theories, however, the pertinent transformations are not the conformal-chiral transformations of the various representations of the Lorentz group but the corresponding transformations for supermultiplets. The result is a modification of the expression for the axial anomaly in the supersymmetry theories.

Let us examine the supermultiplets $\phi_c[A,B]$ containing the fields $[A,B]$ and $[A-\frac{1}{2},B]$, and the "binary" supermultiplets $\phi_c[A,B]$ containing the field $[A,B]$, $[A-\frac{1}{2},B]$, $[A,B-\frac{1}{2}]$ and $[A-\frac{1}{2},B-\frac{1}{2}]$. We define the conformal-chiral transformations for these supermultiplets as follows: $\delta\phi_c[A,B]_c=(\delta f(x)\pm i\delta g(x))\,\phi_c[A,B]_c$ (and an analogous definition for $\phi_c[A,B]$), where the plus sign corresponds to $A\geqslant B$ and the minus sign to $A\leqslant B$. We wish to emphasize that *the sign of $\delta g(x)$ is determined by the leading values of A and B in the given supermultiplet*; this circumstance unambiguously fixes the chiral weights of the various spinor fields and of the $A_{[\mu\nu]}$ fields in the given supermultiplet. For such transformations the conformal-chiral anomaly is, in accordance with (8),

$$(-1)^{2(A+B)}<T^\mu_{\mu\,c}[A,B]>=\alpha_{+\,c}[A,B]\,\frac{1}{32\pi^2}(C_+^2+C_-^2)$$
$$=4^{A+B-1}\left(A-\frac{7}{12}\right)\frac{1}{32\pi^2}(C_+^2+C_-^2), \tag{12}$$

$$(-1)^{2(A+B)}<D^\mu j^5_{\mu\,c}[A,B]>=\alpha_{+\,c}[A,B]\,\frac{1}{32\pi^2}(C_+^2-C_-^2)$$
$$=4^{A+B-1}\left(A-\frac{7}{12}\right)\frac{1}{32\pi^2}(C_+^2-C_-^2) \tag{13}$$

Since

$$\alpha_{-\,c}[A,B]\equiv\alpha_-[A,B]-2\alpha_-[A-\frac{1}{2},B]=0, \tag{14}$$

and

$$<T^\mu_{\mu\,c}[A,B]_c(x)>=<D^\mu j^5_{\mu\,c}[A,B]_c(x)>=0\ , \tag{15}$$

since

$$\alpha_{\pm\, c}\, [A, B]_c \equiv \alpha_\pm\, [A, B] - 2\, \alpha_\pm\, [A - \frac{1}{2}, B] - 2\alpha_\pm\, [A, B - \frac{1}{2}]$$

$$+ 4\alpha_\pm [A - \frac{1}{2}, B - \frac{1}{2}] = 0. \tag{16}$$

It follows from (12) and (14) that for $\phi_c\, [A, B\,]_c$ the conformal and chiral anomalies are determined by the same number, in accordance with (9), while for $\phi_c\, [A, B\,]_c$ both of the anomalies vanish. Expanded supergravities are described in the first loop by means of some set of $N = 1$ superfields $\chi(x, \theta)$, $\Phi_\alpha(x, \theta)$ (chiral) and $H_{\alpha\dot\alpha}(x, \theta, \bar\theta)$, $\Psi_\alpha(x, \theta, \bar\theta)$, $\Psi_\alpha(x, \theta, \bar\theta)$, $V(x, \theta, \bar\theta)$ and (general),[10] which agree in terms of components with the supermultiplets described above:

$\phi_c\, [\frac{1}{2}, 0]$, $2\phi_c\, [1,0]$

and

$\phi_c\, [1,1]_c$, $\phi_c\, [1, \frac{1}{2}]_c$, $\phi_c\, [\frac{1}{2},1]_c$, $\phi_c\, [\frac{1}{2}, \frac{1}{2}]_c$,

respectively. It follows from (13) and (15) that only the fields $\phi_c\, [A, B\,]$ contribute to the axial anomaly in the gravitational field; fields $\phi_c\, [A, B\,]_c$ do not contribute. The single-loop axial anomaly in the expanded supergravity is thus

$$< D^\mu j^5_\mu\, (x) > = [N_{c[\frac{1}{2}, 0]} + 10 N_{2_c[1, 0]}]\, \frac{1}{24 \cdot 32\pi^2}\, (C_+^2 - C_-^2), \tag{17}$$

where $N_{c[1/2,0]}$, $N_{2_c[1,0]}$ is the number of corresponding supermultiplets. The most natural version of supergravity with $N \geqslant 3$ is described only by the fields $\phi_c\, [A, B\,]_c$ (in the superfield description, the corresponding canceling superfield is real[10]), so that both the axial and conformal anomalies are zero for these theories.

We wish to thank V. I. Ogievetskiĭ, A. Tseĭtlin, and A. S. Schwarz for a useful discussion.

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Translated by Dave Parsons
Edited by S. J. Amoretty