Evolution of a soliton under the action of small perturbations

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(Submitted 15 February 1983; resubmitted 3 May 1983)
Pis'ma Zh. Eksp. Teor. Fiz. **37**, No. 11, 542–545 (5 June 1983)

It is shown for the perturbed sine-Gordon equation that the motion of the localized (solitonlike) part of the solution describing the evolution of a soliton under the action of small perturbations does not correspond to the equations of the adiabatic approximation.

PACS numbers: 03.70. + k, 11.10.Qr

Considerable progress has recently been made in the study of nonlinear equations that are exactly integrable by the method of the inverse problem.[1] However, in studying real physical systems, equations arise that as a rule are not exactly integrable. But sometimes the nonlinear equations describing interesting physical phenomena differ little from exactly integrable equations. Solutions of such equations are usually analyzed with the help of perturbation theory.

A well-known nonlinear equation used to describe the properties of various physical systems[2–5] is the sine-Gordon (SG) equation. An equation, which differs little from the SG equation, has the following form in the dimensionless variables:

$$u_{tt} - u_{xx} + \sin u = \epsilon R[u], \tag{1}$$

where $\epsilon = \text{const}$ is a small parameter, while the operator R determines the type of external perturbation. For example, when Eq. (1) describes the dynamics of a one-dimensional crystal in the Frenkel-Kontorov model,[6] the perturbation R takes into account the difference between the equations of the dynamics of a discrete chain and the equation for the leading order continuum approximation. In this case,[5]

$$\epsilon R[u] = \alpha u_x^2 u_{xx} + \beta u_{xxxx}. \tag{2}$$

We shall concentrate on studying the evolution of solitons in systems described by the equation in Ref. 1. It is customarily assumed that the dynamics of the soliton, acted upon by an external perturbation, is described well by the so-called adiabatic approximation.[7–10] However, we shall show that successive application of perturbation theory, based on the method of the inverse problem,[11] gives in first order with respect to the small parameter results differing considerably from the adiabatic approximation.

We seek the solution of Eq. (1) describing the evolution of a soliton under the action of small perturbations in the form[11]

$$u(x, t) = u_s(z) + \epsilon u^{(1)}(x, t) + \dots \tag{3}$$

The first term

$$u_s(z) = 4\,\mathrm{arctg}\; e^z, \qquad z = (x - \zeta)/(1 - v^2)^{1/2}, \tag{4}$$

is called the adiabatic approximation. It coincides in the form with the unperturbed soliton, but its parameters ζ and v depend on time and, in addition, $v = v_0$ and $\zeta = \zeta_0$ at $t = 0$. The functions $u^{(1)}(x,t)$ satisfy the initial conditions $u^{(1)}(x,0) = 0$.

Using the scheme of the scattering problem,[1] as well as the equations describing the evolution of scattering data for the discrete spectrum only under the action of perturbations,[12] we can obtain up to order ϵ equations describing the time dependence of the parameters in the adiabatic approximation

$$\frac{dv}{dt} = -\frac{\epsilon}{4}(1 - v^2)^{3/2} J_0(v), \tag{5}$$

$$\frac{d\zeta}{dt} = v - \frac{\epsilon}{4} v(1 - v^2) J_1(v), \tag{6}$$

where

$$J_n(v) = \int_{-\infty}^{\infty} dz\, \frac{z^n R[u_s(z)]}{\mathrm{ch} z}, \qquad n = 0, 1.$$

Using the equation for the inverse problem[1] and solving the equation of the perturbed evolution of the scattering data for the continuous spectrum,[12] we obtain the first-order correction with respect to ϵ, which we write in the form

$$u^{(1)}(x, t) = u_1(z) + u_2(z, \tilde{z}) + w(z, z^+, z^-), \tag{7}$$

where

$$u_1(z) = \frac{1}{4\mathrm{ch} z}\left\{ \int_{-\infty}^{z} dy\, \frac{R[u_s(y)]}{\mathrm{ch} y} F(z, y) + \int_{z}^{\infty} dy\, \frac{R[u_s(y)]}{\mathrm{ch}\, y} F(-z, -y) \right\}, \tag{8}$$

$$u_2(z, \tilde{z}) = -\frac{v^2 \tilde{z}^2}{4\mathrm{ch} z} J_0(v) + \frac{v^2 \tilde{z}}{2\mathrm{ch} z} J_1(v). \tag{9}$$

Here

$$\tilde{z} = (x - \zeta_0 - t/v_0)/(1 - v_0^2)^{1/2},$$

$$z^{\pm} = (x - \zeta_0 \pm t)/(1 - v_0^2)^{1/2},$$

$$F(z, y) = e^{-z}\mathrm{ch} z + e^y \mathrm{ch} y - z + y + v^2 (z - y)^2 - 1.$$

The last terms in (7) describe the waves emitted by the soliton which propagate with velocities differing greatly from the velocity of the soliton. We do not write out w in explicit form.

We shall examine the odd perturbations for which $R\,[u_s(z)] = -R\,[u_s(-z)]$. Such perturbations include perturbations (12), perturbations of the type examined in Refs. 3 and 4, as well as trivial increments of the type $\alpha \sin u$, which do not spoil the exact integrability of the starting equation.

It is easy to verify that in the case of odd perturbations the second term in (7) and the terms containing v^2 in the term $u_1(z)$ completely compensate the first-order correction, which is obtained from the adiabatic approximation. For this reason, the velocity of the localized part of the solution changes in the initial period for $t \ll t_0 \sim (1-v_0)^{-1/2}$:

$$v = v_0 - \frac{\epsilon v_0 t^4}{6\pi} \int_{-\infty}^{\infty} dz \, \frac{\operatorname{sh} z P[u_s(z)]}{\operatorname{ch}^3 z} . \qquad (10)$$

Thus, under the action of odd perturbations, in contrast to the adiabatic approximation, the initial soliton accelerates according to the law (1).

For $t \gg t_0$, solution (3) in first order with respect to ϵ has the form

$$u_{ds}(x,t) = 4\operatorname{arctg} e^z + \epsilon u_1(z_0), \quad z_0 = (x - \zeta_0 - v_0 t)/(1 - v_0^2)^{1/2}. \qquad (11)$$

We shall call expression (11) a deformed soliton, which is formed by the evolution of the starting localized perturbation. It can be verified that the deformed soliton (11) in the approximation examined is an exact solution in the form of a wave with a stationary profile for the perturbed equation (1). For example, for perturbation (1), the deformed soliton (11) up to terms of first order can be written in the form

$$u_{ds} = 4\operatorname{arc\,tg} \exp\{z[1 - \beta/2(1 - v_0^2)]\} + \frac{(2\alpha - 3\beta)}{(1 - v_0^2)^2} \frac{\left(\frac{2}{3} v^2 z_0 - \operatorname{th} z_0\right)}{\operatorname{ch}^2 z_0} . \qquad (12)$$

For $2\alpha = 3\beta$, expression (11) coincides with the first terms in the expansion of the exact soliton solution found in Ref. 5 in powers of α.

In the case of even perturbations, the motion of the localized part of solution (3) for $t \ll t_0$ is not described by the equations of the adiabatic approximation. For example, under the action of a constant external force, the initial motion of the soliton is not a uniformly accelerated motion, which is well confirmed by numerical experiments.[13]

Thus the description of the motion of a soliton under the action of external perturbations within the scope of the adiabatic approximation, based on examination of the evolution of only the discrete spectrum of the scattering problem, does not describe the physical situation. It is necessary to take into account the corrections that arise due to the evolution of the continuous spectrum under the action of perturbations, which also contribute to the dynamics of the localized part of the solution (deformed soliton).

We thank V. A. Marchenko and the participants of his seminar for useful discussions, as well as A. E. Borovik and B. A. Ivanov for their interest in this work.

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Translated by M. E. Alferieff
Edited by S. J. Amoretty