Localization in an incommensurate potential: exactly solvable multidimensional model

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The one-dimensional model proposed by Grempel, Fishman, and Prange is extended to the multidimensional case. The energy levels and the states which are localized at all energies are found for a broad range of parameters. The conductivity is exponentially small at low frequencies. Localization does not occur if the situation is slightly incommensurate.

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Grempel, Fishman, and Prange[1] recently proposed a model of a one-dimensional discrete incommensurate structure described by the Hamiltonian

$$H^{(1)}_{nn'} = w_{n-n'} + g\,\mathrm{tg}\,(\alpha n + {}^{\nu}/_{2})\delta_{nn'}\,, \tag{1}$$

where w_n is an even-parity, rapidly decaying function of the index of the one-dimensional chain, $g>0$ is a coupling constant, $\alpha/2\pi$ is an irrational number, and $0\leqslant\nu<2\pi$ is the initial phase. Grempel *et al.*[1] found all the energy levels and states for the case in which $\alpha/2\pi$ is not approximated very well by the rational numbers p/q, so that the condition $|\alpha q - 2\pi p| > 2q^{-1}e^{-\gamma q}$, holds, where the definite quantity γ depends on the energy E and the coupling constant g (more on this point below). All the states which they found turned out to be exponentially localized. In this letter we wish to point out that this model has a natural multidimensional analog with the Hamiltonian

$$H^{(d)}_{\mathbf{RR}'} = w_{\mathbf{R}-\mathbf{R}'} + g\,\mathrm{tg}(\alpha_{\mathbf{R}}/2)\,\delta_{\mathbf{RR}'}\,, \qquad \alpha_{\mathbf{R}} = \vec{\alpha}\mathbf{R} + \nu\,, \tag{2}$$

where $\mathbf{R}$ are the sites of the d-dimensional lattice, $w_{\mathbf{R}}$ is an even-parity, rapidly decaying function, ν is the same as in (1), and $\vec{\alpha} = (\alpha_1, \ldots, \alpha_d)$ is a fixed set of 2π—independent numbers, i.e., numbers such that any linear combination of them, $\vec{\alpha}R$ is not a multiple of 2π. It turns out that through elementary algebraic manipulations for the Green's function of the operator $H^{(d)}$ we can derive a convenient representation which we can use for an extremely detailed and exact analysis of the model for any dimensionality (a rather complicated indirect method was used by Grempel *et al.*[1] in their analysis of the one-dimensional case exclusively).

We set $z = E + i\delta$, $G(z) = (H^{(d)} - z)^{-1}$, $G^{(0)}(z) = (\mathbf{W} - z)^{-1}$, $\zeta_{\mathbf{RR}'} = \exp(-i\alpha_{\mathbf{R}})\times\delta_{\mathbf{RR}'}$. It is then simple to prove the following identity:

$$G(z) = (1+\zeta)(1-M\zeta)^{-1}\,G^{(0)}(z+ig),$$
$$M = (ig - z + W)(ig - z - W)^{-1}\,. \tag{3}$$

The operator ζ is unitary, and the doperator M in the momentum representation is multiplication by the function $m(\mathbf{k})$, where $|m(\mathbf{k})| < 1$ at $\delta > 0$. Accordingly, for an arbitrary complex energy the perturbation-theory series in $M\zeta$ converges at the rate of a geometric progression. This convergence is the primary advantage of (3). We will illustrate it in a simple problem, that of calculating the state density $\rho(E)$. For this purpose it is sufficient to find the trace of G, divided by the volume, in the macroscopic limit. We expand the operator $(1 - M\zeta)^{-1}$ in (3) in a series and make use of the independence of the numbers $\alpha_1, \ldots, \alpha_d$; we then easily find that the trace of all the terms of the series containing $\zeta^l, l \neq 0$, makes a vanishing contribution. The unknown quantity is thus simply $G^{(0)}(z + ig)$, and we can write

$$\rho(E) = \pi^{-1}\,\mathrm{Im}\,V^{-1}\,\mathrm{Sp}G^{(0)}(E + ig) = \int \rho_0(E - E')p_C(E')dE', \tag{4}$$

where $\rho_0(E)$ is the state density of the translationally invariant part of (3), and $p_C(x) = g[\pi(g^2 + x^2)]^{-1}$ is the probability density of the Cauchy distribution. In model (2) the state density is therefore the same as in the so-called Lloyd model,[2] in which the role of the potential is played by random quantities $v_{\mathbf{R}}$ which are independent at different values of $\mathbf{R}$, with a Cauchy probability density. It can be shown that the state density is of the form in (4) for Hamiltonian (2) and for any entire series of other incommensurate forms—for a polynomial dependence of α_R on $\mathbf{R}$, for example.

Returning to the case $\alpha_{\mathbf{R}} = \vec{\alpha}\mathbf{R} + \nu$, we note that result (4) could have been derived by integrating the diagonal element $G_{\mathbf{RR}}$ from (3) over ν. This circumstance is a manifestation of the self-averageability property[2] in this model, and we will make use of this simpler averaging procedure. We assume that the numbers $\alpha_1, \ldots, \alpha_d$ are independent to the extent that for all $\mathbf{R}$ and m we can write

$$|\vec{\alpha}\mathbf{R} - 2\pi m| \geqslant A\,|\mathbf{R}|^{-\beta}, \quad \beta > d. \tag{5}$$

Vectors of this type are typical in the sense that the set of these vectors has a complete measure in a d-dimensional space.[3]

We denote by $f_{\mathbf{R}}$ the Fourier coefficient of the function $f(\mathbf{k}) = i\ln m(\mathbf{k})$ and we write it in the form

$$f(\mathbf{k}) = f_0 + t(\mathbf{k}) - t(\mathbf{k} + \vec{\alpha}), \tag{6}$$

where

$$t(\mathbf{k}) = \sum_{\mathbf{R} \neq 0} e^{-i\mathbf{kR}} f_{\mathbf{R}}(1 - e^{-i\vec{\alpha}\mathbf{R}})^{-1}. \tag{7}$$

It can be seen that the coefficients $f_{\mathbf{R}}$ behave in the limit $|\mathbf{R}| \to \infty$ as $R^{-1}G^{(0)}_{\mathbf{R}}(z + ig)$, i.e., fall off exponentially even at real values $z = E$: $f_{\mathbf{R}} \sim \exp(-\gamma R)$. Series (7) thus converges rapidly by virtue of (5), and the function $t(\mathbf{k})$ is analytic at $|\mathrm{Im}k_j| \leqslant \gamma$. From (3) and (6) we find

$$G(z) = (1 + \zeta)e^{it}(1 - e^{-if_0}\zeta)^{-1}e^{-it}G^{(0)}(z + ig). \tag{8}$$

Since the operator $1 - e^{-if_0}\zeta$ is diagonal in the $\mathbf{R}$ representation, we find from (8) that the energy levels E_{nm} of the Hamiltonian are the roots of the equation $f_0 = -\nu + \vec{\alpha}\mathbf{n} + 2\pi m$, and the corresponding states in the $\mathbf{R}$ representation are

$$\psi_{nm}(\mathbf{R}) = \chi(E_{nm}, \mathbf{R}-\mathbf{n}), \qquad (9)$$

where

$$\chi(E, \mathbf{R}) = (2\pi)^{-d} \int_{-\pi}^{\pi} d\mathbf{k} e^{-i\mathbf{k}\mathbf{R} - it}(w - E - ig)^{-1}. \qquad (10)$$

It can be seen that $f_0 = 2\pi N(E)$, where $N(E) = \int_{-\infty}^{E} \rho dE'$. It follows from this relation, the monotonic behavior of $N(E)$, and the independence of the components of $\vec{\alpha}$ that all the levels are nondegenerate, and it follows from (10) and the analytic nature of $t(\mathbf{k})$ that all the states fall off exponentially with distance from the localization center $\mathbf{n}$ in (9) with a decay rate γ. It is also a simple matter to find the density-density correlation function[2] $p(E,\mathbf{R}) = \lim_{\delta \to 0} \delta/\pi |G_{0\mathbf{R}}(E+i\delta)|^2$. If also falls off exponentially as $\mathbf{R} \to \infty$, at a rate 2γ. Finally, we find the following estimate of the conductivity at zero temperature and $\omega \ll E_F$:

$$\sigma(\omega) \leqslant \text{const} \exp\{-A_1(\rho(E_F)/\omega)^{1/\beta}\},$$

where A_1 is of the same order as A in (5).

These properties of model (2) show that there is a pronounced localization in it at any energy. In disordered systems, this localization usually occurs only in the fluctuation part of the spectrum.[2] This circumstance is quite understandable in the one-dimensional case, since the potential in (1) takes on arbitrarily large but finite values at a series of points running off to infinity. Interestingly, even in the multidimensional case the "forest" of these peaks gets so thick that it also causes an exponential localization of the states at all energies. We wish to emphasize that all these results hold only if the numbers α are "sufficiently irrational." In this case we could also have a function $\exp(-\gamma' R)$ with $\gamma' < \gamma$ on the left in (5). If, however, $|\vec{\alpha}\mathbf{R} - 2\pi m| < A \exp(-BR^{1+\epsilon})$, $\epsilon > 0$ then we can show through the use of (3) that the $H^{(d)}$ spectrum contains not a single bound state (in the mathematical literature a spectrum of this sort is called a "singularly continuous" spectrum). These cases, however, should be regarded as exceptional, since the set of the corresponding $\vec{\alpha}$ has a zero measure.[3]

[1]D. R. Grempel, S. Fishman, and R. E. Prange, Phys. Rev. Lett. **49**, 833 (1982).
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