Phase transition in the 2*D* Ising model with impurity bonds

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The problem of critical behavior of the 2*D* Ising model with impurity bonds is solved by using the Feynman-walk method in the 2*D* lattice. The fluctuational part of specific heat was found to be equal to $C_{fl} \sim -\ln\ln|\tau|$ as $\tau \to 0$, in contrast to the pure Ising model for which $C_{fl} \sim -\ln|\tau|$.

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Second-order phase transitions in inhomogeneous physical systems are currently the subject of intensive investigation. It is now clear that the impurities do not generally eliminate the singularities in the thermodynamic functions.[1] However, the defects can have a strong influence on the singularities, i.e., they can change the critical indices. This effect was demonstrated in Ref. 1 using the ϕ^4 model. Attempts have also been made to explain the influence of impurities on the phase transition for the simple ferromagnetism model—the Ising model (IM). Here, the 2*D* model, for which there is an exact analytical solution in the absence of impurities,[2] is of interest in itself. Several important facts have been established concerning the phase-transition line (in the T plane, and ν is the density of impurity bonds), see, for example, Ref. 3. In particular, the slope angle of the transition line as $\nu \to 0$ and the critical impurity concentration ν_c at which the phase transition disappears $T_c(\nu_c) = 0$ (Ref. 5) are known for the model with broken bonds (defect bond $\tilde{J} = 0$).[4] However, its critical behavior as $T \to T_c(\nu)$ has not been investigated.

We present in this paper the results of an exact solution of the critical behavior of the specific heat of the 2*D* Ising model with a small density ($\nu \ll 1$) of the broken bonds $\tilde{J} \neq J$. It turns out that the critical behavior of the model for $\tau \equiv (T - T_c)/T_c \to 0$ varies in a universal manner; namely, the fluctuational part of the specific heat behaves in the following manner:

$$C_{fl} \sim -\ln|\tau| \quad \text{for} \quad \tau >> \tau_\nu, \tag{1}$$

$$C_{fl} \sim -\ln\ln|\tau| \quad \text{for} \quad \tau << \tau_\nu, \tag{2}$$

where $\tau_\nu \sim \exp\{-\text{const}/\nu\}$ is the temperature scale in which the critical behavior changes—a transition from the pure Ising model to a disordered model. We note that, although the small ν is used in the solution, the universality of the critical behavior gives us serious reasons to assume that the fluctuational part of the specific heat of the disordered Ising model is defined by Eq. (2) as $T \to T_c(\nu)$ at any density $\nu < \nu_c$.

We shall examine the 2*D* Ising model with impurity bonds. Its statistical summation has the form

$$Z = \sum_{\{\sigma\}} \exp\{ \sum_{x,\alpha} K_{x,\alpha} \sigma_x \sigma_{x+\hat{\alpha}} \}. \tag{3}$$

Here $K_{x,\alpha} \equiv J_{x,\alpha}/T$ has the value $\beta = J/T$ in most bonds, and the broken $\beta = \tilde{J}/T$ in the impurities which are randomly distributed in the lattice with a small density ν. Calculation of the statistical sum (3) by using the standard method reduces to a summation over the configurations of the closed loops and then to a Feynman walk through the lattice with the phase $\exp(i\Delta\phi/2)$ for each turn of the wandering particle.[6] After these transformations, the statistical sum (3) assumes the form $Z = \exp - F$

$$F = \sum_P (\prod_P \lambda_i) \Phi(P). \tag{4}$$

Here $\lambda_i \equiv \lambda_{x,\alpha} = \tanh K_{x,\alpha}$ is the scalar weight of one bond spacing (x,α), $\prod \lambda_i$ is the scalar weight of the closed path P, and $\Phi(P) = \prod \exp\Phi(i\Delta\phi/2)$ is the phase factor of the amplitude of the path P. We must average the free energy (4) over the random coefficients λ_i. The averaging for each lattice bond (but not the path) in our model is done independently: $\lambda_i = \lambda$ with a probability of $1 - \nu$, and $\tilde{\lambda}$ with a probability ν. It is important that the paths, in which one lattice bond can be traversed several times, contribute to the summation over the paths for the free energy (4). For example, there is no shortcut (becasue of the amplitude phase factor) for the path in Fig. 1, since the sum in (4) is taken over single loops. As a result, the averaging of the free energy (4) results in the "cementing" of paths at the recurrent bonds, and there will no longer be graphs for the free walks.

The problem is solved in the following manner. It is known that the ordinary $2D$ Ising model in the critical region is equivalent to the free-fermion model. In the Lagrangian approach used by us this means that the 4-component (in a square lattice) walking objects (see Ref. 6) are transformed into 2-component objects in the critical region. In other words, only two of the four degrees of freedom fluctuate strongly and are described by a free spinor field (in the Euclidean formulation) with a mass $m \sim \tau$:

$$A_0[\psi] = \int d^2x \, (\bar{\psi} \hat{\partial} \psi + m\bar{\psi}\psi). \tag{5}$$

If the impurity concentration is small, then initially the fluctuations will develop as in the pure Ising model, as τ decreases. With further decrease of τ and increase of the correlation length $r_c \sim 1/\tau$, the influence of the impurities becomes important. As

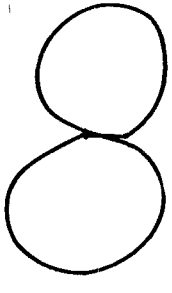

FIG. 1.

pointed out, the averaging free energy (4) results in a "cementing" of the wandering-particle paths. It follows from the arguments presented above that the spinor field describes the disordered Ising model in the critical region ($\tau \to 0$) in an equivalent manner, but this time with an interaction. The interaction in this case can be easily determined. In fact, the randomness of the lattice bonds is simulated in a certain sense by the randomness of the mass in Eq. (5). A Gaussian averaging over the masses gives the Lagrangian

$$A[\psi] = \int d^2x \left[\bar{\psi}^a \hat{\partial} \psi^a + m_0 \bar{\psi}^a \psi^a - g_0 (\bar{\psi}^a \psi^a)^2\right]. \tag{6}$$

Here $m_0 = \langle m \rangle$. Notice that these are only guiding considerations. In fact, all four degrees of freedom of the lattice wandering object, rather than just the strongly fluctuating ones, which are described by the Lagrangian (5), contribute equally to the exact numerical parameters m_0 and g_0 (see below) in Eq. (6). The spinor field ψ^α is an N-component (Grossman) variable, and it is assumed that $N = 0$ in the results. The well-known De Gennes trick, which is used in polymer theory for summing over single paths,[7] is used here. We reduce in our model the sum over the single paths for the free energy to the zero-component Lagrangian theory.

The exact solution of the problem in the critical region is as follows. The free energy (4) is averaged. We obtain a sum over the paths, in which the recurrent bonds are included with an additional weighting. We must go over the continuous limit for this sum and then compare the plots with the diagrams of the Lagrangian model (6) for $N = 0$. We shall describe the conversion to continuous limit in a later, more detailed paper. The conclusion, however, is that the disordered Ising model in the critical region is equivalent to the Lagrangian model (6) (for the formal value $N = 0$) with the parameters

$$m_0 = 2\frac{\lambda_c - \lambda}{\lambda_c^0} \approx c_1 \tau, \qquad g_0 = \nu \frac{a^2(1 + 2b(1 + ab))}{(1 + ab)^2} = c_2 \nu. \tag{7}$$

Here $\lambda = \text{th}\beta$, $\lambda_c^0 = \sqrt{2} - 1$, $a = (\text{th}\tilde{\beta} - \lambda_c^0)/\lambda_c^0$, $b = (\sqrt{2})\lambda_c^0/4$, and c_1 and c_2 are numbers ~ 1,

$$\lambda_c \equiv \text{th}\,\beta_c = \lambda_c^0 - \nu \frac{\lambda_c^0 a}{1 + ab} \tag{8}$$

(all the calculations are accurate to the first power of the impurity concentration ν). The Lagrangian (6) with the constants (7) describes the critical behavior of the model on the high-temperature side ($\tau > 0$). The following substitutions must be made for the low-temperature phase: $\lambda \to \lambda^* = \exp(-2\beta)$, $a \to a^* = (\exp(-2\tilde{\beta}) - \lambda_c^0)/\lambda_c^0$:

$$\lambda_c \to \lambda_c^* \equiv \exp(-2\beta_c) = \lambda_c^0 - \nu \frac{\lambda_c^0 a^*}{1 + a^* b}. \tag{8*}$$

We note that Eq. (8) gives the well-known numerical value[3,4] in the special case of the broken-bond model ($\tilde{\beta} = 0$)

$$\frac{d\,\mathrm{th}\,\beta_c}{d\nu} = \frac{\lambda_c^0}{1-b} = 6\sqrt{2} - 8.$$

The next solution poses no problem. The model (6)–the Gross-Neveu model[8]—is renormalizable. The charge and mass renormalization equations have the form

$$\frac{dg}{d\xi} = -2\frac{1-N}{\pi}g^2, \qquad \frac{d\ln m}{d\xi} = -\frac{1-2N}{\pi}g. \tag{9}$$

Here $\xi = \ln(1/p)$. See, for example, Refs. 9 and 10 for renormalization-group methods. For $N = 0$ we obtain

$$g(\xi) = \frac{g_0}{1 + \frac{2g_0}{\pi}\xi}, \qquad m(\xi) = \frac{m_0}{\sqrt{1 + \frac{2g_0}{\pi}\xi}}. \tag{10}$$

We obtain for the specific heat (compare Ref. 10)

$$C_{\text{fl}} \approx -\frac{c_1^2}{2}\int\frac{d^2p}{(2\pi)^2}\left(\frac{m(p)}{m_0}\right)^2 Tr\,G^2(p) \approx \frac{c_1^2}{4c_2\nu}\ln\left(1 + \frac{2c_2\nu}{\pi}\ln\frac{1}{c_1 r}\right). \tag{11}$$

Here $G(p) = (m_0 + i\hat{p})/(m_0^2 + p^2)$ is the spinor Green's function.

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