Grassmann analyticity and extension of supersymmetry

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The concept of analyticity of Grassmann spinor variables is introduced. It is shown that this concept makes it possible to realize $N = 2$ supersymmetry on the ordinary, complex, $N = 1$ superpole outside the mass shell.

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1. The Majorana spinor is regarded as a real Grassmann variable in the theory of supersymmetries

$$\Theta_\alpha = \begin{pmatrix} \Theta_\alpha \\ \bar{\Theta}^{\dot{\alpha}} \end{pmatrix}, \qquad \Theta = C\bar{\Theta}^T, \tag{1}$$

where Θ_α and $\bar{\Theta}^{\dot{\alpha}}$ are adjoint Weyl spinors and C is the charge-conjugation matrix. A direct approach in the extended N supersymmetries gives N Grassmann variables. A simple superfield $\Phi(x,\Theta^1,\Theta^2,..\Theta^N)$ contains 2^{4N} field components.

We shall introduce the concept of Grassmann analyticity and demonstrate that the number of Grassmann variables can be reduced with its help by using a simple example of $N=2$ supersymmetry. The Cauchy-Riemann analyticity condition

$$\frac{\partial}{\partial z} f(x, y) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f(x, y) = 0$$

has the following analog for Grassmann variables:

$$\left(\tilde{D}^{\Theta}_{\alpha} + i\tilde{D}^{\eta}_{\alpha}\right)\Phi(x, \Theta, \eta) = 0, \tag{2}$$

where $\tilde{D}^{\Theta}_{\alpha}$ ($\tilde{D}^{\eta}_{\alpha}$) is a covariant spinor derivative of the Grassmann variable Θ (η). This condition, which represents the superpole's "independence" of the Θ-$i\eta$ variable, makes it possible to turn to the complex scalar superpole of a single Grassman variable $V(x,\Theta)$. We should point out that the condition (2) was used in the two-dimensional case in the theory of supersymmetric strings.[1] Further, the chirality is explained the same way as analyticity in the ordinary $N=1$ supersymmetry. In fact, the chirality

$$\bar{D}_{\dot{\alpha}}\Psi(x, \Theta) = 0 \tag{3}$$

means that $\Psi(x,\Theta)$ depends only on the left-handed Weyl spinor Θ^α and is independent of the right-handed adjoint spinor $\bar{\Theta}^\alpha$. The solution of Eq. (3) (see, for example, Ref. 2) is

$$\Psi(x, \Theta) = a(x_L) + \Theta^a\Psi_a(x_L) + \Theta^a\Theta_a b\,(x_L), \qquad x_L^a = x^a + \frac{1}{2}\,\bar{\Theta}\gamma^a\gamma_5\Theta\,. \tag{4}$$

The solution of the analyticity condition (2) given below will be used for reducing the superpole of the two Majorana Grassmann variables to the superpole defined by a single Grassmann variable.

2. We proceed to our problem. We arrived at Grassmann analyticity by starting from the known fact that the complexified $N=1$ supermultiplets in the mass shell are $N=2$-supersymmetry representations with a central charge (see, for example, Ref. 3). This also seems to apply to superfields outside the mass shell. We now consider the complex superfield $V(x,\Theta)$. Its expansion in the terms

$$V(x, \Theta) = \frac{1}{m}M(x) - i\bar{\Theta}\frac{\Psi^2}{m}(x) + \frac{1}{2}\bar{\Theta}\Theta\frac{P^{12}}{m}(x) + \frac{1}{2}\bar{\Theta}\gamma_5\Theta S(x)$$

$$+ \frac{1}{2}\bar{\Theta}i\gamma^a\gamma_5\Theta V_a(x) + \bar{\Theta}\Theta\bar{\Theta}\left[\Psi^1(x) - \frac{1}{2m}\partial\!\!\!/\Psi^2(x)\right] + \frac{1}{4}(\bar{\Theta}\Theta)^2$$

$$\times\left[iP^{11}(x) - \left(\frac{\Box}{2m} + m\right)M(x)\right] \tag{5}$$

includes the complex vector and scalar fields $V_a(x)$, $M(x)$, and $S(x)$, which produce the 0(2) singlets; $P^{ij}(x)$ is a complex, spurless, symmetric tensor 0(2) and $\Psi^i(x)$–0(2) is a doublet of the Dirac spinors. The action for $V(x,\Theta)$

$$S = \int d^4x d^4\Theta\frac{1}{2}\;V^*(x, \Theta)\left[\Box + \frac{(\bar{D}D)^2}{16} + m^2\right]V(x, \Theta) = \tag{6a}$$

$$= \int d^4x\left\{-\frac{1}{2}F^*_{ab}(x)F^{ab}(x) + m^2\overset{*}{V_a}(x)V^a(x) + \partial_a M^*(x)\partial^a M(x) - m^2M^*(x)M(x)\right.$$

$$\left. + m^2S^*(x)S(x) + \frac{1}{2}P^{ij*}(x)P^{ij}(x) + i\bar{\Psi}^k(x)\partial\!\!\!/\Psi^k(x) + im\epsilon^{kl}\bar{\Psi}^k(x)\Psi^l(x)\right\} \tag{6b}$$

is invariant with respect to the 0(2)-supersymmetry transformations

$$\delta V_a = \bar{\epsilon}^k i\gamma_a\gamma_5\Psi^k + \frac{i}{m}\epsilon^{kl}\bar{\epsilon}^k\gamma_5\partial_a\Psi^l,$$

$$\delta\Psi^i = iP^{ij}\epsilon^j + \left(-mM + \frac{i}{2}\sigma_{ab}\gamma_5F^{ab}\right)\epsilon^i$$

$$+ \epsilon^{ik}(-im\gamma_5S + m\gamma^a\gamma_5V_a - \partial\!\!\!/M)\epsilon^k \quad , \tag{7a}$$

$$\delta M = - i\epsilon^{ij}\bar{\epsilon}^i\Psi^j, \quad \delta S = \bar{\epsilon}^i\gamma_5\Psi^i - \epsilon^{kl}\bar{\epsilon}^k\gamma_5\frac{\not\partial}{m}\Psi^l,$$

$$\delta P^{ij} = -\bar{\epsilon}^i(\not\partial\Psi^j + m\Psi^k\epsilon^{jk}) + \frac{1}{2}\delta^{ij}\bar{\epsilon}^l(\not\partial\Psi^l + m\Psi^k\epsilon^{lk}) + (i\leftrightarrow j). \quad (7b)$$

$F_{ab} = \partial_a V_b - \partial_b V_a$, $\epsilon^{12} = -\epsilon^{21} = 1 = \not\partial = \partial^m\gamma_m$ in these formulas and the indices of the internal symmetry were intentionally written on one level in order to emphasize that we are now dealing with 0(2) rather than with $SU(2)$. The parameter of the ordinary supersymmetry is represented by ϵ^1 and the parameter of the second supersymmetry is ϵ^2. These transformations are written in the superfield form as follows:

$$\delta V = - i\bar{\epsilon}^i Q^i V$$

moreover,

$$Q^1_{\bar\beta} = i\frac{\partial}{\partial\bar\Theta^\beta} - (\not\partial\Theta)_\beta, \quad (8a)$$

$$Q^2_\beta = -2m\Theta_\beta + \frac{1}{4m}\bar{D}DD_\beta; \qquad \left(D_\beta = \frac{\partial}{\partial\bar\Theta^\beta} - i(\not\partial\Theta)_\beta\right) \quad (8b)$$

and

$$\{\bar{Q}^i, Q^k\} = 2\delta^{ik}\gamma^a P_a + 2i\epsilon^{ik}Z. \quad (9)$$

It is important that our 0(2) superalgebra has a central charge Z proportional to the mass

$$ZV = mV, \qquad ZV^* = -mV^*. \quad (10)$$

Like the electric charge, it has opposite values for the particles and anitparticles. It is somewhat surprising that there is a term with three spinor derivatives in Eq. (8b). We shall show that these transformations occur naturally.

3. We shall analyze the complex superfield $\Phi(x,\Theta,\eta)$, which satisfies the Cauchy-Riemann condition (2), i.e., it is analytic. The supersymmetry generators for it, which are defined in the form

$$Q^1_a = i\frac{\partial}{\partial\bar\Theta^a} - (\not\partial\Theta)_a, \quad Q^2_a = i\frac{\partial}{\partial\bar\eta^a} - (\not\partial\eta)_a - 2\Theta_a Z \quad (11)$$

satisfy the commutation relations (9). The influence of Z on Φ and Φ^* is defined as in (10) with the substitution of Φ for V. Thus, the operators of the spinor derivatives in the condition (2) can be written as follows:

$$\tilde{D}^\Theta_a = \frac{\partial}{\partial\bar\Theta^a} - i(\not\partial\Theta)_a + 2i\eta_a Z, \qquad \tilde{D}^\eta_a = \frac{\partial}{\partial\bar\eta^a} - i(\not\partial\eta)_a. \quad (12)$$

The Cauchy-Riemann condition can be resolved [analog (4)]:

$$\Phi(x, \Theta, \eta) = e^{-m\bar{\eta}\eta} \phi(x^m - \bar{\Theta}\gamma^{m\eta}, \Theta + i\eta) \tag{13a}$$

$$= e^{-m\bar{\eta}\eta} e^{-\bar{\Theta}\not{\partial}\eta} e^{i\bar{\eta}\frac{\partial}{\partial\bar{\Theta}}} \phi(x, \Theta) . \tag{13b}$$

It appears that the superfield $V(x,\Theta)$ introduced above can be expressed in terms of $\phi(x,\Theta)$ in the following way:

$$V(x, \Theta) = \phi(x, \Theta) + \frac{1}{4m} \bar{D}D\phi(x, \Theta) . \tag{14}$$

Thus,

$$\delta\Phi(x, \Theta, \eta) = -i\epsilon^i Q^i \Phi(x, \Theta, \eta) \tag{15}$$

correspond exactly to the transformations (8) for $V(x,\Theta)$. It is interesting that the action (6) can be written in the form

$$S = \frac{1}{32} \int d^4x d^4\Theta d^4\eta \, \Phi^*(x, \Theta, \eta) \Phi(x, \Theta, \eta) \tag{16a}$$

$$= \frac{1}{2} \int d^4x d^4\Theta \, \phi^*(x, \Theta) \left[\Box + \frac{(\bar{D}D)^2}{8} + \frac{m}{2}\bar{D}D + m^2\right] \phi(x, \Theta)$$

$$= \frac{1}{2} \int d^4x d^4\Theta \, V^*(x, \Theta) \left[\Box + \frac{(\bar{D}D)^2}{16} + m^2\right] V(x, \Theta) . \tag{16b}$$

The representation (16) demonstrates the existence of a strong analogy with the ordinary chiral Lagrangian.

4. The representation (13a) suggests a presence of "analytic" and "antianalytic" bases

$$x^m - \bar{\Theta}\gamma^m\eta, \quad \Theta + i\eta, \tag{17a}$$

$$x^m + \bar{\Theta}\gamma^m\eta, \quad \Theta - i\eta, \tag{17b}$$

which resemble the right-hand and the left-hand bases in $N = 1$ (Ref. 4) and which are transformed into them as $\eta \to y_5^{\Theta}$ [with an appropriate change in the x scale because of differentiation of Eq. (2). In fact, (17a) and (17b) for invariant spaces of the $N = 2$ supersymmetry. It is expected that $N = 2$ supergravitation, which does not contain external constraints and is analogous to that proposed for $N = 1$ in Ref. 4, can be formulated. To do this, 0(2) must first be expanded to $SU(2)$ and the analytic properties of the supermultiplets in Ref. 5 must be analyzed. We can assume that there is an internal coupling with the systems of hypercomplex numbers in the case of higher supersymmetries.[6]

We have verified that the hypercomplex coordinates

$$\tilde{x}^m = x^m + i\bar{\Theta}\gamma^m \eta_k e_k - \frac{i}{2}\bar{\eta}_k \gamma^m \eta_l f_{klp} e_p, \quad \tilde{\Theta} = \Theta + e_k \eta_k$$

$$(e_k e_l = -\delta_{kl} + f_{klp} e_p) \quad (18)$$

form invariant spaces of the $N = 4$ supersymmetry (e_k are the quaternion units) and of the $N = 8$ supersymmetry (e_k are the octave units). These problems will be discussed in our future papers.

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[1]M. Ademollo, L. Brink *et al.*, Nucl. Phys. **B111**, 77 (1976).
[2]A. Salam and J. Strathdee, Phys. Rev. **11D**, 1521 (1975); V. Ogievetskiĭ and L. Mezinchesku, Usp. Fiz. Nauk **117**, 637 (1975) [Sov. Phys. Uspekhi **18**, 960 (1975)].
[3]P. Fayet, Preprint LPTENS 80/7.
[4]V. Ogievetskiĭ and E. Sokatchev, Phys. Lett. **79B**, 222 (1978).
[5]P. Breitenlohner and M. F. Sohnius, Nucl. Phys. **B165**, 483 (1980); M. F. Sohnius, K. S. Stelle, and P. C. West, Preprint ICTP/79-80/51.
[6]J. Lukierski, Preprint UGVA-DPT 1979/09-216.

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