Instability of a charged surface of liquid helium

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It is shown that the main nonlinearity of a charged surface is associated with the invariance of the sign of the charge density in the limit of low surface charge density. The development of an instability with the increase of the pressing field is analyzed qualitatively. The parameters of a single, many-electron dimple are calculated.

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The formation of a lattice of dimples on a charged helium surface has recently been observed experimentally.[1] We shall describe qualitatively the development of such instability and calculate the parameters of a single dimple. We shall use the following expression for the energy

$$\mathscr{E}=\frac{1}{2}\int[(\nabla\xi)^2+\xi^2]\,d^2r-E\int n\xi\,d^2r+\frac{1}{2}\int\frac{n(\mathbf{r})\,n(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^2r\,d^2r', \quad (1)$$

in which the surface tension, gravitation, and interaction of electrons with the external field E and with each other are taken into account. The vertical displacement of the surface $\xi(\mathbf{r})$ and the two-dimensioanl coordinate $\mathbf{r}$ were measured in units of $(\alpha/\rho g)^{1/2}$, the field E and the charge density $n(\mathbf{r})$, which are assumed to be positive, are measured in $(\rho g\alpha)^{1/4}$, and the energy $\mathscr{E}$ is measured in $\alpha^2/\rho g$, where α is the surface tension coefficient, ρ is the helium density, and g is the acceleration of gravity.

The uniform distribution of charge $n(\mathbf{r})=n\sim$ for the field $E_c=(4\pi)^{1/2}$ in the low density limit $n(\mathbf{r})\ll 1$ is unstable for the charge-density wave (CDW) with a unit wave vector of the modulus.[2] The energy $\mathscr{E}$ near the instability threshold is related to the amplitude $n\sim$ of the CDW by the relation

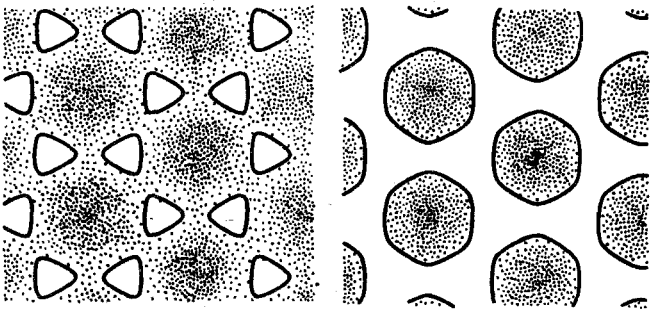
$$\mathscr{E}\backsim(E_c-E)\tilde{n}^2. \quad (2)$$

If $E<E_c$, then the minimum of (2) will be reached at $n\sim=0$. If, however, $E>E_c$, then the energy will decrease with increasing $n\sim$, and the nonlinear constraints on the value of $n\sim$ must be taken into account in order to stabilize the CDW.

The new mechanism for nonlinear charged surface proposed by us is based on the condition that the sign of the charge density $n(\mathbf{r})\geqslant 0$ must be constant. We shall illustrate the effect of this mechanism by using the CDW with a hexagonal symmetry

$$n(\mathbf{r})=\bar{n}+\tilde{n}[\cos x+\cos(x/2+\sqrt{3}\,y/2)+\cos(x/2-\sqrt{3}\,y/2)]. \quad (3)$$

This type of CDW (with $n\sim>0$) was predicted by Gor'kov and Chernikova[2] and



a b

FIG. 1. Distribution of the charge at the helium surface for (a) $E<E_c^f$ and (b) $E>E_c^f$. The solid line separates the regions with $n(r)>0$ from those with $n(r)=0$.

observed experimentally by Leĭderer and Wanner.[1] The expression (3) is positive everywhere, provided that $\tilde{n}<2\bar{n}/3$. This means that a CDW is produced with an amplitude $\tilde{n}_0=2\bar{n}/3$ as a result of discontinuous passage of the field E through the value E_c, consistent with Eq. (3); moreover, $n(\mathbf{r})$ vanishes at the honeycomb hexagonal lattice sites. As the field increases, these sites expand into regularly spaced islands in which $n(\mathbf{r})=0$ (see Fig. 1a). The period of the structure in this case varies, giving an energy minimum (1).

The islands link up at a certain field E_c^f and the regions with $n(\mathbf{r})>0$ turn out to be isolated from each other; therefore, the charges of these regions remain the same with further increase of the field. Since the average charge density is specified, the period of the structure also remains constant, retaining the value reached at $E=E_c^f$. As the field increases, the size of the regions occupied by the electrons decreases (see Fig. 1b) and we obtain a triangular lattice of dimples with a Coulomb interaction within the range of a strong field.

As shown in Ref. 2, the nonlinear corrections to Eq. (1) are of order $\bar{n}^4$. A comparison with Eq. (2) shows, therefore, that they change the picture described above only near the instability threshold when $|E-E_c|\sim\bar{n}^2$. Notice that the nonlinearity proposed by us is substantial within this field range. This nonlinearity limits the magnitude of the discontinuity $\tilde{n}$ to $2\bar{n}/3$ at the threshold, which is approximately fivefold smaller than the discontinuity $3\bar{n}/(63/8-4\sqrt{3})$ obtained in Ref. 2. The hysteresis effects predicted in Ref. 2 also decrease accordingly (see also Ref. 3).

We shall now introduce the equations that determine $\xi(\mathbf{r})$ and $n(\mathbf{r})$. To do this, we must add the term $-\lambda\int n(\mathbf{r})d^2r$ to the energy (1) (the selection of the Lagrangian multipliers λ ensures that the total charge at the surface remains invariant) and vary the obtained functional in $\xi(r)$ and $n(r)$. Finally, we obtain

$$-\Delta\xi(\mathbf{r})+\xi(\mathbf{r})=En(\mathbf{r}); \tag{4}$$

$$-E\,\xi(\mathbf{r})+\int\frac{n(\mathbf{r}')d^2r'}{|\mathbf{r}-\mathbf{r}'|}=\lambda. \tag{5}$$

Because of the condition $n(\mathbf{r})\geqslant 0$, the last equation is valid only in that region of $\mathbf{r}$

in which $n(\mathbf{r})>0$. We can see from Eq. (5) that the value of λ is equal to the total electrostatic potential in the regions occupied by electrons. In the more general case, another parameter λ should be introduced for each region with $n(\mathbf{r})>0$; these parameters must be determined from the specified charges of the regions. If there are time variations, then Eq. (4) must be supplemented by a $\xi(\mathbf{r})$ time derivative and the parameter λ in Eq. (5) must be considered as a function of time. Note that the linear equations (4) and (5) generally have many solutions. The correct solution must be selected from the minimum condition of (1) in conjunction with the condition $n(\mathbf{r})\geqslant 0$.

We used Eqs. (4) and (5) in the calculation of a single dimple (such dimples were observed by Leiderer and Wanner[1] in their experiment).

We can assume that $n(\mathbf{r})>0$ for a single dimple in a circle of radius R. It is convenient to use the following expansions inside the circle:

$$\xi(r)=\sum_{m=0}^{M}\Sigma_m(r/R)^{2m}; \tag{6}$$

$$n(r)=\sum_{m=0}^{M}N_m(1-r^2/R^2)^{(m-1/2)}. \tag{7}$$

Expansion (6) and its first derivative join with the function $\xi(r)\sim K_0(r)$ a$_+$ $r=R$ (K_0 is the Bessel function). The expansion (7) was normalized to the total charge Q of the dimple. Taking this into account, we calculated the energy (1) in the quadratic form of Σ_m and N_m with R-dependent coefficients. Minimization of $\mathscr{E}$ required the solution of the system of linear equations for $\Sigma_m(R)$ and $N_m(R)$ and a subsequent minimization of R. It was shown that the minimum of $\mathscr{E}$ is reached when $N_0(R)=0$, so that we have $n(r)\sim(1-r^2/R^2)^{1/2}$ as $r\to R$.

The calculation showed that a dimple can exist at $E>E_D=3.698$. This field determines the stability boundary of the metastable states of a dimple, since $\mathscr{E}(E_D)>0$, i.e., it exceeds the energy of the unperturbed surface. As the field increases, $\mathscr{E}(E)$ decreases and passes through zero at $E=4.062$. $\mathscr{E}\approx -0.159\,Q^2E^2\,ln(0.224\,E)$ in the range of large fields.

The charge-density distribution for $E=E_D$ and in the limit $E\to\infty$ is shown in Fig. 2.[1)]

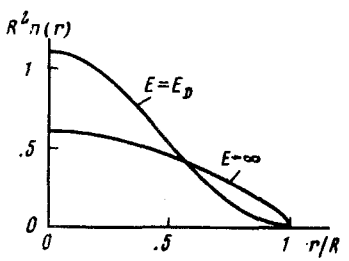


FIG. 2. Distribution of the charge of a hole for $E=E_D$ and for $E\to\infty$.

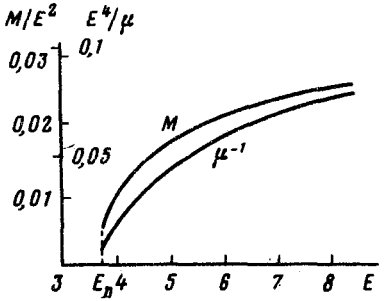


FIG. 3. Dependence of the mass and mobility of a hole on the applied field.

The hydrodynamic mass of a dimple and its mobility, which are expressed in units of $Q^2\rho/\alpha$ and $\alpha^2/Q^2\rho^2 g\nu$ (ν is the helium viscosity), are described by the expressions

$$M = \frac{1}{8\pi}\int_0^\infty \xi_k^2\, k^2 dk; \qquad \mu = \frac{\pi}{2}\left[\int_0^\infty \xi_k^2\, k^4 dk\right]^{-1}, \tag{8}$$

where ξ_k is the Fourier transform of the $\xi(r)$ function. The field dependence of the mass and mobility is shown in Fig. 3. $M \approx E^2/32$ and $\mu \approx 9.9\, E^{-4}$ within the range of a strong field.

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[1)]The $n(r)$ function for $E = E_D$ is very similar to the Gaussian function. The use of the Gaussian trial function for $n(r)$ (this approximation was proposed by Leiderer and Shikin[4]) gives the value 3.711 for E_D, which differs by only 0.3% from the exact value.

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