Whistler leakage from plasma waveguides

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Whistlers cannot be completely confined in a high-density plasma waveguide because of tunneling. The wave damping rate corresponding to leakage from the waveguide is derived. The results explain several experimental facts.

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1. The waveguide propagation of whistlers has been under intense study in recent years, primarily because of whistler self-focusing and the hypothesis of magnetospheric ducts: channels with a high plasma density in which whistlers bounce between conjugate points in the geomagnetic field.

An interesting question which has not yet been answered is why, in the case of pronounced self-focusing of whistlers, one always observes waveguides with a density lower than that in the surrounding plasma and never with a higher density,[1] although the theory based on the nonlinear Schrödinger equation predicts both possibilities.[2–4]

In this letter we show that, in contradiction of the results that follow from geometric optics and from the Schrödinger equation, whistlers always leak out of high-density waveguides, explaining the results reported by Stenzel.[1] The equations derived below give a quantitative description of this leakage and lead to a natural explanation for several effects that have been observed in other experiments.[5,6]

2. We assume that an external magnetic field is directed along the z axis, that the electron gyrofrequency ω_H is constant, and that the two-dimensional model is valid. We assume that the electron plasma frequency ω_p varies along x. Considering only a single component of the wave field, E_y (for brevity), we can write

$$E_y = iF(x)\exp[i(\omega/c)pz - i\omega t]\,. \tag{1}$$

If the width of the waveguide is much larger than c/ω, we can use the WKB approximation. We find four independent solutions from Maxwell's equations under the conditions $(m_e/m_i)^{1/2}\omega_H \ll \omega < \omega_H \ll \omega_p$. For $F(x)$ in (1) these solutions are

$$F_k^{\pm}(x) = q_k^{-1/2}(p^2 - 4a)^{-1/4}r_k \exp\left(\pm i\frac{\omega}{c}\int_0^x q_k dx\right), \tag{2}$$

where $k = 1,2$; $a(x) = \omega_p^2(x)/\omega_H^2$; $r_k^2 = p - (-1)^k(p^2 - 4a)^{1/2}$;

$$q_k^2 = \frac{1}{2u^2}[(1 - 2u^2)p^2 - 2a + (-1)^k p(p^2 - 4a)^{1/2}]\,; \tag{3}$$

and $u = \omega/\omega_H$. The subscript "k" specifies one of the two whistler branches. The important distinction between these two branches is that the x projection of the energy flux has a sign which is the same as that of q_k in the case $k=1$ and opposite in the case $k=2$.

It follows from (3) that if $u<1/2$ then $q_2^2>0$ for $a<p^2/4$ and $q_1^2\geqslant 0$ for $a_0\leqslant a<p^2/4$, where $a_0=p^2u(1-u)$. For $u>1/2$ we always have $q_1^2<0$, and we have $q_2^2\geqslant 0$ for $a\leqslant a_0$.

We assume $u<1/2$ and that the density profile $a(x)$ has a symmetric hump: $a(x)=a(-x)$. Then under the condition

$$\max\left\{4a(0),\ \frac{a(\infty)}{u(1-u)}\right\} < p^2 < \frac{a(0)}{u(1-u)} \tag{4}$$

a plot of $q_k(x)$ would take the form in Fig. 1. It follows that a waveguide (in the geometric-optics sense) forms for branch q_1. In this case $a(\pm x_0)=a$. The branches q_1 and q_2 are isolated by virtue of the condition $a(0)<p^2/4$ [see (4)], but deviations from geometric optics disrupt this isolation; this effect can be seen in the conversion $q_1\to q_2$, which results in the leakage of energy from the waveguide. To study this process we write $F(x)$ in the form

$$F(x) = C_1F_1^+(x) + C_2F_1^-(x) \quad (-x_0 < x < x_0), \tag{5}$$

$$F(x) = A_1|q_1|^{-1/2}(p^2-4a)^{-1/4}\, r^1\exp\left(-\frac{\omega}{c}\left|\int_{x_0}^{x} q_1\,dx\right|\right) + A_2F_2^- \quad (x\to\infty), \tag{6}$$

$$F(x) = B_1|q_1|^{1/2}(p^2-4a)^{-1/4}\, r^1\exp\left(-\frac{\omega}{c}\left|\int_{-x_0}^{x} q_1\,dx\right|\right) + B_2F_2^+ \quad (x\to-\infty). \tag{7}$$

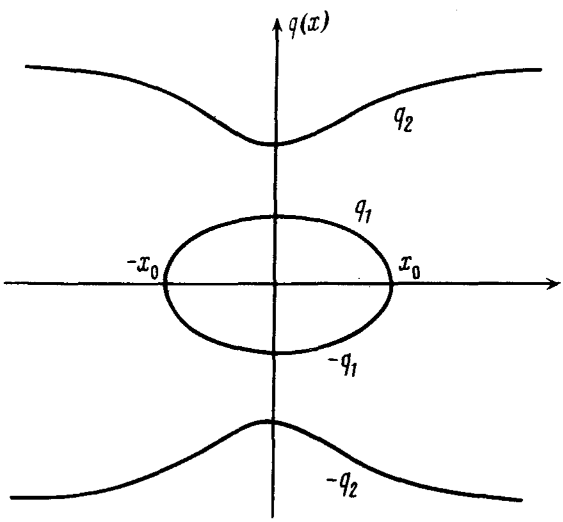

FIG. 1.

The terms $A_2F_2^-$ and $B_2F_2^+$ in (6) and (7) describe the leakage. These terms are small under conditions such that the WKB approximation is legitimate. The leakage of a wave results in its damping along the z axis, i.e., in the appearance of a positive imaginary part of p and thus of q_k, as can be seen from (1). Assigning a superscript 0 to quantities which correspond to the case without leakage, we can write $p=p^{(0)}+ip'$, $q_1=q_1^{(0)}+iq'_1$, and

$$q_1' \approx \frac{\partial q_1^{(0)}}{\partial p}p' = -\frac{v_{gz}(q_1^{(0)})}{v_{gx}(q_1^{(0)})}\frac{c}{\omega}\mu \tag{8}$$

(v_g is the group velocity), where we have used the relation $\partial_q/\partial_p=-v_{gz}/v_{gx}$, which follows from the dispersion relation and we have introduced the wave damping rate $\mu=(\omega/c)p'$, which can be determined by using the relationship between the coefficients in (5)–(7).

For this purpose we continue (6) onto the interval $|x|<x_0$ and then into the region $x<-x_0$, and we compare the results with (5) and (7). Since the points $\pm x_0$ are singularities, they must be circumvented in the complex plane. The other singularities, which are important in this problem, are the points at which $q_1=q_2$, i.e., at which $a(x)=p^2/4$. It can be seen Fig. 1 that these points lie in the complex plane. It can be shown that for a wide variety of symmetric profiles the singularities of this type, which are nearest the real axis, are arranged symmetrically on the imaginary axis, at $x=\pm i\xi$, $\xi>0$. Considering paths which run above and below the singularities $\pm x_0$, $\pm i\xi$, we find

$$|A_2/C_2|^2 = |B_2/C_1|^2 = T, \tag{9}$$

$$T = \exp\left\{-2\frac{\omega}{c}\,\mathrm{Im}\int_0^{i\xi}[q_2^{(0)} - q_1^{(0)}]dx\right\}. \tag{10}$$

Here T is the coefficient of the conversion $q_1\to q_2$, which we may call a tunneling, since in this case the path crosses the complex plane. Equation (10) is analogous to that derived in Ref. 7 for the conversion which occurs when a whistler propagates across a transition region with slowly varying properties. It was also shown in Ref. 7 that the condition $T\ll 1$ holds if the WKB approximation is valid.

Ignoring the terms with A_2 and B_2 in (6) and (7), we find the usual quantization condition

$$\frac{\omega}{c}\int_{-x_0}^{x_0} q_1^{(0)}\,dx = \pi\left(n+\frac{1}{2}\right), \quad n = 0, \pm 1, \pm 2, \ldots.$$

To determine μ we write the conditions for conservation of the energy flux at the points $\pm x_0$ for the case with leakage:

$$|C_1|^2\exp\left(-2\frac{\omega}{c}\int_0^{x_0} q_1'\,dx\right) - |C_2|^2\exp\left(\frac{2\omega}{c}\int_0^{x} q_1'\,dx\right) = |A_2|^2,$$
$$|C_2|^2\exp\left(2\frac{\omega}{c}\int_0^{-x_0} q_1'\,dx\right) - |C_2|^2\exp\left(-2\frac{\omega}{c}\int_0^{-x_0} q_1'\,dx\right) = |B_2|^2 \qquad (11)$$

(Since A_2 and B_2 are small, we have omitted the exponential functions with q_2' from the right sides.) Using (8) and (9) we find the following result from (11) in first order in q_1':

$$\mu = \frac{1}{2}\left(\int_{-x_0}^{x_0} \frac{v_{gz}(x)}{v_{gx}(x)}\,dx\right)^{-1} T. \qquad (12)$$

Equations (9), (10), and (12) constitute the complete solution of the problem.

In the experimental results displayed in Ref. 5 we can clearly see wave damping under these conditions. Leakage has been observed directly by Balmashnov.[6]

Waveguide propagation cannot occur in channels with a high density if $u \geqslant 1/2$; low-density channels may be ideal waveguides for both $u \leqslant 1/2$ and $u > 1/2$.

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