One- and two-loop invariants in expanded supergravitations

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(Submitted 15 January 1981)

Pis'ma Zh. Eksp. Teor. Fiz. **33**, No. 5, 292–295 (5 March 1981)

It is shown that the finiteness of the first two loops in $N = 8$ supergravitation follows from supersymmetry and it follows in the $N = 2$ and $N = 4$ theories on condition that chiral-dual invariance is satisfied.

PACS numbers: 11.30.Rd, 11.30.Pb

1. The question of one- and two- loop renormalizability of the $N = 2,..., 8$ expanded supergravitations,[1] on which the main hope for constructing a unified theory of all fundamental interactions is now bared, has not been investigated until now, although it was hoped that an increase in N can improve the situation. As is known, the $N = 1$ theory is finite in the first two loops.[1-3] In this theory there is only one superinvariant counterterm in the first loop, which unifies $(R_{\mu\nu\alpha\beta})^2$ and vanishes in the mass shell (MS). Although only the $(R_{\mu\nu\alpha\beta})^3$ term can appear in the second loop, it is forbidden by supersymmetry. There is a candidate for the counterterms in the third loop.[3]

2. We shall analyze the superinvariant counterterms in the first two loops in the expanded supergravitations by using our results of Ref. 4, where the linearized superfields in the MS are described for $N = 2,..., 8$ and the exact superinvariants, beginning with the 8th loop[2]) and the linearized, three-loop invariant in the $N = 8$ theory are given. In this analysis we make use of the fact that the dimensionality of dx is equal to -1, the dimensionality of $d\theta$ is equal to $\frac{1}{2}$, and the dimensionality of the gravitational constant k is equal to -1. The dimensionality of investigated superfields will be indicated everywhere henceforth.

A convenient method of correlating superinvariants to the number of loops is to use only the superfields from Refs. 4 and 5, in which fields of all spins are normalized by means of the gravitational constant k, so that the quantity k no longer enters into the superfield. In this case the boson fields Φ have zero dimensionality, while the spinor fields have a dimensionality of $\frac{1}{2}$. By means of these fields the entire original Lagrangian of the expanded supergravitation is represented in the form $(1/k^2)$ $L(\Phi,\chi_\alpha,..., g_{\mu\nu})$. The number of loops l is this case is equal to the number of propagators n, from which the number of vertices m is subtracted, and the *entire* dependence of the counterterms on k^2 has the form $k^{2(n-m+1)}$, i.e., $k^{2(l-1)}$ [even when the corresponding invariant does not contain a purely gravitational term of the type $(R_{abcd})^{l+1}$.

3. We investigate the one-loop and two-loop counterterms in the $N = 2$ supergravitation, where the one-loop counterterms were analyzed in the component formalism and the coefficients in front of them were calculated first in Ref. 1. The counterterm $R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}$ was discussed in this paper ($R_{\mu\nu\alpha\beta}$ is the curvature and $F^{\mu\nu}$ is the intensity of the vector field) and it was assumed that this term is forbidden by the dual

invariance. In addition, the coefficient for the counterterm $T^2_{\mu\nu} = (F_{\mu\mu}{}' F_{\mu'\nu} - \frac{1}{4} g_{\mu\nu}{}' F^2_{\mu'\nu'})^2$ was calculated and it was found to be equal to zero.

4. In the superfield formalism in the MS the $N = 2$ theory is described in a linearized approximation by a chiral superfield with unit dimensionality[4,5]:

$$W_{ab} = F_{ab} + \theta_i^c \Psi^i_{abc} + \frac{1}{2}\theta_i^c \theta_j^d \epsilon^{ij} R_{abcd},$$
$$\overline{W}_{\dot a \dot b} = \overline{F}_{\dot a \dot b} + \overline{\theta}^{\dot c i} \overline{\Psi}_{i\, \dot a \dot b \dot c} + \frac{1}{2}\overline{\theta}^{\dot c i} \overline{\theta}^{\dot d j} \epsilon_{ij} \overline{R}_{\dot a \dot b \dot c \dot d}, \quad (1)$$

where a and b are the two-component spinor indices and F, Ψ, and R are the intensities of the fields for spins 1, 3/2, and 2, respectively. In the one-loop approximation only one invariant of zero degree in k^2 can be formulated from this superfield

$$S^{l=1}_{N=2} = \int d^4x\, d^4\theta\, W_{ab} W_{ab} + \text{H.c.} = \int d^4x\, (R^2_{abcd} + \overline{R}^2_{\dot a \dot b \dot c \dot d}) + \dots, \quad (2)$$

which vanishes in the MS. The invariant $R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}$, as we see from the superfield formalism, is not forbidden by the duality, but simply by the supersymmetry, which also forbids the above-mentioned $T^2_{\mu\nu}$ term. The nontrivial linearized invariant

$$S^{l=2}_{N=2} = k^2 \int d^4 s\, d^4\theta\, (W_{ab} W_{ab})^2 + \text{H.c.} = k^2 \int d^4x\, [F^2_{ab} R^2_{cdel} + \overline{F}^2_{\dot a \dot b} \overline{R}^2_{\dot c \dot d \dot e \dot l}] + \dots \quad (3)$$

exists in the two-loop approximation. This invariant contains no purely gravitational part. It is possible, however, that it has no nonlinear unification. We note also that in Eq. (3) there is an integral over the left-hand superspace of only the left-hand chiral superfields. We analyze it from the viewpoint of chiral-dual invariance.[6] A chiral-dual invariance in the MS in the linearized-superfield approach means that the counterterms must not depend on a in the transfromation $d\theta_{ai} \to a^{-1/2} d\theta_{ai}$, $d\overline{\theta}_{i\dot a} \to a^{1/2} d\overline{\theta}_{i\dot a}$, $W_{ab} \to a W_{ab}$, and $\overline{W}_{\dot a \dot b} \to a^{-1} \overline{W}_{\dot a \dot b}$. The two-loop invariant (3) is forbidden by the above-indicated chiral-dual invariance. We emphasize, however, that the linearized supersymmetry allows this invariant and that the question of finiteness of the $N = 2$ theory in the second loop is attributable to the less reliable (in the sense of possible anomalies) chiral-dual invariance.

As is known, the supersymmetrical invariant exists in the three-loop approximation in the $N = 2$ theory.[7]

4. In $N = 3$ theory, in the MS there is a linearized chiral spinor superfield with a dimensionality of $\frac{1}{2}$:

$$W_a = \chi_a + \theta_i^b F^i_{ab} + \frac{1}{2!}\theta_i^b \theta_j^c \epsilon^{ijk} \Psi_{k\,abc} + \frac{1}{3!}\theta_i^b \theta_j^c \theta_k^d \epsilon^{ijk} R_{abcd}$$

from which it is impossible to construct superinvariants—candidates as the counterterm in the first and second loop; however, the three-loop invariant exists.[4,5]

In $N = 4$ theory, in the MS there is a linearized chiral scalar superfield of zero dimensionality:

$$W = \Phi + \theta_i^a \chi_a^i + \frac{1}{2!} \theta_i^a \theta_j^b F_{ab}^{ij} + \frac{1}{3!} \theta_i^a \theta_j^b \theta_k^c \epsilon^{ijkl} \Psi_{labc}$$

$$+ \frac{1}{4!} \theta_i^a \theta_j^b \theta_k^c \theta_l^d \epsilon^{ijkl} R_{abcd} \cdot$$

In the one-loop approximation we can construct the following candidate as the counterterm:

$$S_{N=4}^{l=1} = \int d^4x d^8\theta W^4 + \text{H.c.} = \int d^4x \Phi^2 R_{abcd}^2 + \dots$$

in addition to the normal term that vanishes in the MS and unifies $(R_{\mu\nu\alpha\beta})^2$. This invariant, as can be fully shown, can be forbidden either by using chiral dual invariance or by specifying that it has no unification. However, just as in the case of the second loop in $N = 2$, the linearized supersymmetry allows it.

5. We proceed now to the most interesting case of $N = 8$ theory. This theory is described in the linearized approximation in the MS by the scalar superfield W_{ijkl} of zero dimensionality.[4,5] As shown by the author of Ref. 4, the superfield W_{1234} in the "self" basis depends only on the "domestic" Grassman variables, θ_1, θ_2, θ_3, and θ_4 and on the "foreign" variables $\bar\theta^5$, $\bar\theta^6$, $\bar\theta^7$, and $\bar\theta^8$, and an analogous picture exists for the remaining components of the field W_{ijkl}. This field, therefore, depends on the 16 Grassman variables (instead of the 32 variables $\theta_{i\alpha}$, $\bar\theta^i_{\dot\alpha}$, $i = 1,\dots, 8$), and this makes it possible to write the linearized superinvariant in $N = 8$ theory, which is the minimum superinvariant with respect to the number of loops, in the form

$$S_{N=8}^{l=3} = k^4 \int d^4x (d^8\theta \, d^8\bar\theta)_{\text{eigen}} \; W_{1234}^4 = k^4 \int d^4x (R_{abcd} \bar R_{\dot a\dot b\dot c\dot d})^2 + \dots,$$

and it is also an $SU(8)$ invariant.[4] From the fields of spins 0,..., 2 that exist in $N = 8$ theory in the mass cloud, it is impossible to construct other superfields, which would give linearized invariants in the first two loops. It is interesting that a one-loop term of the type (2), which also vanishes in the mass cloud but which is not equal to zero in topologically nontrivial fields, cannot constructed in the $N = 8$ theory [this may be an additional indication (see Ref. 8) that the conformal supergravitation is missing at $N > 4$]. It is known, on the other hand, that the absence of counterterms in the linearized approximation is a sufficient condition for the absence of superinvariants.[3] Thus, the $N = 8$ supergravitations is finite in the first two loops.

The author is grateful to V. I. Ogievetskiĭ and E. S. Fradkin for a discussion of the questions raised in this paper.

[1)]The quantity N is the number of graduated parameters of the algebra, or the gravitinos (field with spin 3/2).

[2)]The linearized superfields in the MS for $2 \leqslant N \leqslant 8$ and the exact 8-loop invariants in the $N = 8$ theory were obtained independently in Howe and Lindström's paper.[5]

[1]M. T. Grisaru, P. van Nieuwenhuizen, and J. A. M. Vermaseren, Phys. Rev. Lett. **37**, 1662 (1976).
[2]M. T. Grisaru, Phys. Lett. B **66**, 75 (1977).
[3]S. Deser, J. H. Kay, and K. Stelle, Phys. Rev. Lett. **38**, 527 (1977).

[4]R. E. Kallosh, Lebedev Phys. Inst. Preprint No. 152, 1980; Phys. Lett. (in press).
[5]P. Howe and U. Lindström, Preprint Ref. TH. 2953-CERN 1980.
[6]P. van Nieuwenhuizen and J. A. M. Vermaseren, Phys. Lett. B **65**, 263 (1976); S. Ferrara, J. Scherk, and B. Zumino, Nucl. Phys. B **121**, 393 (1977).
[7]S. Deser and J. H. Kay, Phys. Lett. B **76**, 400 (1980).
[8]B. de Wit and S. Ferrara, Phys. Lett. B **81**, 317 (1979).

Translated by Eugene R. Heath
Edited by S. J. Amoretty