Alternative calculation of the instanton contribution to the β function in the Yang-Mills theory

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The leading contribution of instantons to the charge renormalization was calculated. The β function was observed to increase rapidly in the region $g \approx 1$, in good agreement with the extrapolation of the results for strong coupling in the lattice.

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The interpolation of the β function between the weak- and strong- coupling regimes[1,2] in the Yang-Mills $[Su(N)]$ theory is of interest currently. Callan, Dashen, and Gross[3] (CDG) have thus far proposed the only mechanism which allows one to go outside the context of the conventional perturbation theory, if only for small g, within the framework of a continual approach. Their analysis, however, contains rather artificial components from the point of view of the standard theory: "instanton medium" and "vacuum penetrability" μ_{vac} are used for multiplicative field and charge renormalization of the form

$$g_{ren}(1/a) = \mu_{vac}(\rho_c)\, g^2(1/a), g^2(1/a) = \frac{8\pi^2}{b_N \ln(1/a\Lambda)}, \quad b_N = \frac{11N}{3} \quad . \qquad (1)$$

where ρ_c is the infrared cutoff of the instanton size ρ and a is the lattice constant of the effective lattice theory with the coupling constant $g_{ren}(1/a)$. Arguments have been given in favor of identification of $\rho_c \approx a$ in order to obtain a single dimensional CDG parameter. The extent to which Eq. (1) reflects the renormalization-group properties, which are known in the context of perturbation theory, is not clear, irrespective of the possibility of justifying such an equation.

We propose in this letter a different, more common and direct method of estimating the instanton contributions to the β function, i. e., analysis of the leading radiative and quasi-classical corrections in the three- and two-point Green's functions in the Yang-Mills theory. We shall use the rarefied instanton gas approximation[4] (RGA) and ignore the (dipole) interactions between the instantons. The contribution to the Green's functions in the leading order in g_0^2 in the instanton sector comes only from the classical fields $A^{inst} \sim 0(1/g_0)$ with a weight equal to the single-instanton amplitude $d(\rho)$ in the one-loop approximation[5]

$$d(\rho) = C_N x_0^{2N} e^{-x_0} (M\rho)^{b_N} e^{-A(\bar{\rho})\rho^2} \rho^{-4}, \quad x_0 = \frac{8\pi^2}{g_0^2} \quad . \qquad (2)$$

where g_0 is the renormalized charge and M is the Pauli-Willars regularization parameter. Exponential cutoff corresponds to the hard core which accounts for the gas rarefaction and which was proposed in Ref. 6; $A(\bar{\rho}) = a_N/\bar{\rho}^2$, where $a_N = (b_N - 4)/2$ and $\bar{\rho}$ is the average size of instantons, which is uniquely connected with the rarefaction parameter a'; this parameter controls the small interaction between the two (anti) instantons, $|x_1 - x_2|^4/\rho_1^2\rho_2^2 > a'$. An estimate of the parameter a' from the condition of positive action gives[6] $a' \geqslant 0\,(100)$.

Using the Fourier transform of the instanton in the singular gauge, we obtain an unrenormalized, untruncated, three-point function at the symmetric point $k_i k_j = -k^2\, 1 - 3\delta/2$ (in the Landau gauge)

$$G^{a_1 a_2 a_3}_{\mu_1\mu_2\mu_3}(k_1, k_2, k_3)|_{s.p.} = i(2\pi)^4\delta(\Sigma k_i)\left[1 + \frac{61N}{12}x_o^{-1}\ln\frac{M}{k} + \frac{\pi^2 C_N}{N(N^2-1)}\right.$$

$$\left.\times\, x_o^{2N+2}e^{-x_o}\left(\frac{M}{k}\right)^{b_N} I(k\bar{\rho})(1 + 0(g_o^2))\right]\frac{1}{k^6}\Gamma^{(o)\,a_1 a_2 a_3}_{\mu_1\mu_2\mu_3}(k_1, k_2, k_3)$$

$$+\ kkk\ \text{terms.} \tag{3}$$

Here we have used the $\Gamma^{(0)}$ notations for the bare, three-gluon vertex and

$$I(k\bar{\rho}) = 64\int\frac{d\rho'}{\rho'}\rho'^{\,b_N - 4}\left(1 - \frac{\rho'^2}{2}K_2(\rho')\right)^3\exp\left[-a_N\left(\frac{\rho'}{k\bar{\rho}}\right)^2\right]$$

and also have taken into account the diverging part of the one-loop contribution of the conventional perturbation theory. The leading contribution to the gluon propagation is calculated in the same manner.

The minimum-subtraction procedure, in which the Z constants contain only the logarithmically diverging parts, can be used for renormalization of the Green's functions. The instanton contributions are also well defined after charge renormalization, but the corresponding $\beta_{\min}$ coincides with the conventional, one-loop expression.[7] We shall use a different renormalization scheme, i. e., the momentum-subtraction scheme and require that the renormalized functions must be normalized at a certain point $k^2 = \mu^2$ to the Born term with a substitution of the renormalized coupling constant for the bare coupling constant. In this case Eq. (3) makes it possible to determine the instanton contribution to the constants $Z_3^3 Z_1^{-1}$. Finally, after taking into acount the renormalization constant of the propagator Z_3, we have

$$g_{ren} = Z_3^{3/2} Z_1^{-1}\left(\frac{M}{\mu}, \mu\bar{\rho}\right)g_o = g_o\left\{1 + \frac{b_N}{2}x_o^{-1}\ln\frac{M}{\mu} + 0(g_o^4)\right.$$

$$\left. + \frac{\pi^2 C_N}{N(N^2-1)}x_o^{2N+2}e^{-x_o}(M/\mu)^{b_N} I(\mu\rho)(1 + 0(g_o^2))\right\}. \tag{4}$$

The instanton contribution to Z_3 is missing in Eq. (4), since it has been suppressed

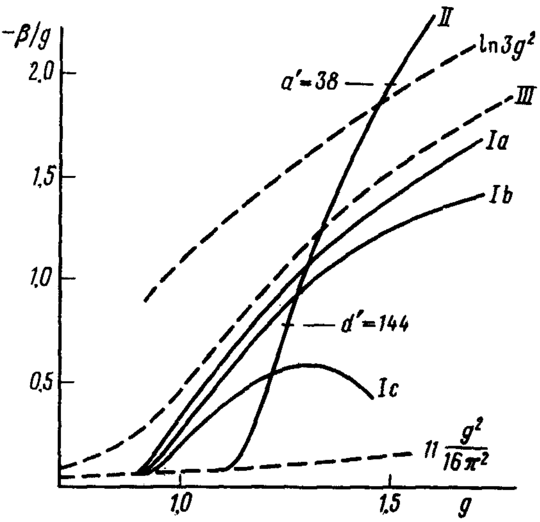

FIG. 1. The β function in $SU(3)$ gauge theory with allowance for instanton contributions. I. Pulse subtraction circuit for different resolutions: (a) $a'=30, \bar{\rho}\Lambda_{\rm latt}=0.0055$; (b) $a'=114, \bar{\rho}\Lambda_{\rm latt}=0.0048$; (c) $a'=691, \bar{\rho}\Lambda_{\rm latt}=0.0040$. II. CDG renormalization[3] [Eq. (1)] with $a=\bar{\rho}(a')$ (Ref. 6) (ignoring instanton interaction). III. Padé extrapolation of the strong-coupling expansion in a Euclidean lattice.[1] The dashed curves represent the leading strong- and weak-coupling expansion terms, respectively.

as $0(g_0^2)$.

Using the expression for a single-loop running coupling constant $g(\mu)$, we obtain a β function as follows:

$$\beta = g_0 \mu \frac{\partial}{\partial \mu} Z_3^{3/2} Z_1^{-1} = \beta(g(\mu), \bar{\rho}\Lambda), \quad g_{ren} = g_{ren}(g(\mu), \bar{\rho}\Lambda).$$

The product $\bar{\rho}(a')\Lambda$ appears here as a free parameter, which is constrained only by the gas-rarefaction criterion mentioned above.

To compare the obtained results with the Euclidean-lattice calculations, we must select an appropriate regularization scheme, which is equivalent to varying the parameter Λ and the general constant C_N: $\Lambda_{PV}/\Lambda_{\rm latt}=31.3$ (Ref. 8), $C_N^{PV}/C_N^{\rm latt}=(\Lambda_{PV}/\Lambda_{\rm latt})^{-b}N$. Figure 1 illustrates the $\bar{\beta}$ function for several values of the parameter a', which was obtained in this manner. The curve with $a'=114$ is acceptable, whereas $a'=30$ lies in the region in which RGA applies to a lesser extent.[6]

Two distinguishing features can be pointed out: detachment from a one-loop β function occurs at $g\approx 0.9$ almost independently of the degree of rarefaction of the gas; the slope of the curve corresponds to the Padé extrapolation of the $\bar{\beta}$ function for strong-coupling regime.[1]

Our renormalization scheme conceptually is quite remote from the CDG procedure[3] whose result[6] is given in Fig. 1 for comparison. According to the comparison (1), the gas rarefaction varies with the distance from the strong-coupling curve. The instanton gas, nonetheless, is sufficiently rarefied to preclude any hope that instanton interactions would guarantee a smooth transition to the strong-coupling regime. We pointed out in another paper[6] that the curves must cross in the CDG renormalization scheme. We have used this to determine the maximum space-time occupied by instantons, $f \approx 0.01$, placing our hope on another mechanism which may suddenly come into effect. Our formulation, however, has no crossing, and the rarefied gas, of course, describes the entire transitional region quite satisfactorily. The question whether allowance for instanton interactions and for higher-order corrections with respect to g^2 will destroy the obtained attractive picture needs further study.

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