Infinite set of conservation laws for a relativistic string

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A solution of the Cauchy problem is found and an infinite set of conserved values for a free, closed, relativistic string is constructed.

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The main achievement in the recent development of quantum-field methods can apparently be considered the exact solution of nonlinear, two-dimensional models which have a remarkable property: an infinite number of conservation laws.[1–4] In view of this, the search for new systems, which also have this property (of course, this does not imply that the model is completely integrable) is of particular interest.

We shall obtain in this study an infinite set of nontrivial, conserved quantities for a free, closed, relativistic string. As is well known,[5] this system is one with the constraints: $\chi_a = 0$,

$$\chi_0 = \frac{1}{2}\left(m^2 x'_\mu x'^\mu + \frac{1}{m^2} p_\mu p^\mu\right), \quad \chi_1 = x'_\mu p^\mu, \quad x'_\mu = \frac{\partial}{\partial s} x_\mu \tag{1}$$

where $x_\mu(\tau, s) = x_\mu(\tau, s+2\pi)$ is a world surface swept over by a closed string in a four-dimensional space-time, $p_\mu(\tau, s)$ is the pulse density at the s point of a string at the time τ, and m is a certain parameter of the mass dimension. A linear combination of χ_0 and χ_1 [the algebra of the constraints (1) guarantees the absence of secondary constraints] plays the role of the Hamiltonian for the systems with the constraints (1),[6]

$$\mathcal{H}(\tau) = \int ds\, (V_0(\tau, s)\chi_0 + V_1(\tau, s)\chi_1) = \chi_0(V_0) + \chi_1(V_1) \tag{2}$$

where V_0 and V_1 are arbitrary functions whose specific choice is equivalent to gauge fixing. We shall write the solution of the Hamiltonian equations of motion in the form

$$p_\mu(\tau, s) = U(0, \tau) * p_\mu(s), \quad x_\mu(\tau, s) = U(0, \tau) * x_\mu(s); \tag{3}$$

here $p_\mu(s)$ and $x_\mu(s)$ are the reference data and $U(0, \tau)$ is the evolution operator

$$U(0, \tau) = T \exp \int_0^\tau dt\, \mathcal{H}(t)$$

$$\exp\left(A * B\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \{A, \{A, \{A, \ldots\ldots\{A, B\}\}\ldots\}, \tag{4}$$

where $\{A, B\} = \int ds(\delta A/\delta x_\mu\, \delta B/\delta p^\mu - \delta A/\delta p_\mu\, \delta B/\delta x^\mu)$ are the Poisson brackets. The Poisson brackets on the right-hand side of Eq. (3) can be calculated explicitly. To do this, we need only to mention that the algebra of constraints (1) is a direct sum of two algebras, each of which is isomorphous with respect to the overparametrization algebra (Virasoro algebra). In fact, we shall select those which are formed in the space of the constraints (1) in the form

$$M(f) = X_1(f) + X_0(f) = \int ds\, f(s) a^2(\tau, s),$$
$$K(f) = X_0(f) - X_1(f) = \int ds\, f(s) b^2(\tau, s). \tag{5}$$

Here we introduce new variables[1)]

$$a_\mu = \frac{1}{\sqrt{2}}\left(\frac{1}{m} p_\mu + m x'_\mu\right), \quad b_\mu = \frac{1}{\sqrt{2}}\left(\frac{1}{m} p_\mu - m x'_\mu\right), \quad \{a, b\} = 0. \tag{6}$$

We obtain the following relations for the values M and K:

$$\{M(f), M(g)\} = M(g'f - f'g), \{M(f), K(g)\} = 0$$
$$\{K(f), K(g)\} = K(g'f - f'g), \quad f' = \frac{\partial}{\partial s} f, \quad g' = \frac{\partial}{\partial s} g \tag{7}$$

which proves the assumption given above. Usings Eqs. (2), (5), (6), and (7), we obtain

$$a_\mu(\tau, s) = U(0, \tau) * a_\mu(s) = a_\mu(\Phi(\tau, s)) \frac{\partial}{\partial s} \Phi,$$
$$b_\mu(\tau, s) = U(0, \tau) * b_\mu(s) = b_\mu(\Gamma(\tau, s)) \frac{\partial}{\partial s} \Gamma, \tag{8}$$

where $a_\mu(s)$ and $b_\mu(s)$ are the input data and the V_0, V_1, Φ, and Γ functions are related by the relations

$$\frac{\partial \Phi}{\partial \tau} = +(V_0 + V_1) \frac{\partial \Phi}{\partial s}, \quad \frac{\partial \Gamma}{\partial \tau} = -(V_0 - V_1) \frac{\partial \Gamma}{\partial s}. \tag{9}$$

Recalling the definition (8) for the variables a_μ and b_μ, we can rewrite the system (3) in the form

$$U(0, \tau) * p_\mu(s) = \frac{1}{2}\left[p_\mu(\Phi) \frac{\partial \Phi}{\partial s} + p_\mu(\Gamma) \frac{\partial \Gamma}{\partial s}\right] + \frac{1}{2}\left[\frac{\partial}{\partial s} x_\mu(\Phi) - \frac{\partial}{\partial s} x_\mu(\Gamma)\right]$$
$$U(0, \tau) * x_\mu(s) = \frac{1}{2} \int_\Gamma^\Phi p_\mu(\lambda)\, d\lambda + \frac{1}{2}(x_\mu(\Phi) + x_\mu(\Gamma)). \tag{10}$$

In fact, Eq. (10) is the general solution of the Cauchy problem.[2)] The Φ and Γ functions are related to the V_0 and V_1 functions by the relations (9). Using this solution we can construct three quantities (which depend on two arguments) in such a way that the transformation (3) reduces for them simply to overparametrization

$$\frac{1}{\sqrt{2}} A_\mu^1 = \int_{s_2}^{s_1} a_\mu(\tau, s)\, ds, \quad \frac{1}{\sqrt{2}} A_\mu^2 = \int_{s_2}^{s_1} b_\mu(\tau, s)\, ds$$

$$A_\mu^3 = \int_{s_2}^{s_1} p_\mu(\tau, s)\, ds + x_\mu(\tau, s_1) + x_\mu(\tau, s_2); \tag{11}$$

We can clearly see now that any parametrically invariant functional J constructed from the quantities A^1, A^2, and A^3 will be conserved $[U(0, \tau\, {}^*J = J$ or $\{\mathscr{H}, J\} = 0]$. We look for J in the form

$$J_1(q^\mu) = \int ds_1 ds_2 a_\mu(\tau, s_1)\, a_\nu(\tau, s_2) \exp iq^\mu A_\mu^1,$$

$$J_2(q^\mu) = \int ds_1 ds_2\, b_\mu(\tau, s_1)\, b_\nu(\tau, s_2) \exp i q^\mu A_\mu^2, \tag{12}$$

$$J_3(q^\mu) = \int ds_1 ds_2\, a_\mu(\tau, s_1)\, b_\nu(\tau, s_2) \exp iq^\mu A_\mu^3.$$

Here q^μ is a four-vector. To satisfy the relation $\{J_\alpha, \mathscr{H}\} = \partial J_\alpha/\partial\tau = 0$, it is necessary and sufficient to impose the following condition on q_μ:

$$P_\mu q^\mu = 2\pi n, \tag{13}$$

where $P_\mu = \int_0^{2\pi} p_\mu(s, \tau) ds$ is the total momentum and n is an integer. The condition (13) is a periodicity condition of the expressions under the integral sign for J_α. Thus, J_α are the three-parametric generating functions of the conservation laws. We shall write several basic conservation laws.

a) Total-momentum P conservation law

$$J_\mu = \int ds\, a_\mu(s) = \frac{1}{\sqrt{2}} P_\mu, \quad J_\alpha^{\mu\nu}(q = 0) = \frac{1}{2} P^\mu P^\nu,$$

b) Angular-Momentum conservation law

$$\frac{\partial}{\partial q_\mu} J_{3\mu\nu}(q, n = 0)\Big|_{q=0} = \{\text{with allowance for } qP = 0\} = M_{\mu\nu} P_\mu$$

(there is no summation over μ).

c)

$$\frac{\partial}{\partial q_\mu} (J_1^{\nu\rho} + J_2^{\nu\rho})\Big|_{\substack{n=0\\ q=0}} = \{\text{with allowance for } qP = 0\} = P^\nu t^{\rho\nu\mu}$$

(there is no summation over ν).

Here $t_{\rho\nu\mu} = P_\rho \int ds(\psi_\nu' \psi_\mu + x_\nu' x_\mu) +$ (cyclic permutation of $\rho\nu\mu$), and

$$\psi_\mu' = \frac{\partial}{\partial s} \psi_\mu = p_\mu(\tau, s) - P_\mu / 2\pi.$$

In conclusion, we note that the quantities $J_\alpha(q)$ cannot be automatically carried over to the quantum case since its expressions under the integral sign have products of any number of operators at one point. To give meaning to the quantum values J_α, we must order their expressions under the integral sign without violating the commutability with the Hamiltonian. We can see from the special examples (a), (b), and (c) that this can be done.

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[1)]The mass parameter m, which can always be reconstructed by means of an appropriate gauge transformation, will be dropped henceforth.

[2)]The solution of the Cauchy problem for an infinite string in the gauge $V_0 = 1$, $V_1 = 0$ was obtained in Ref. 7.

1. V. E. Korepin and L. D. Faddeev, Teor. Mat. Fiz. **25**, 143 (1975).
2. A. B. Zamolodchikov, Pis'ma Zh. Eksp. Teor. Fiz. **25**, 499 (1977). [JETP Lett. **25**, 468 (1977)].
3. A. B. Zamolodchikov and Al. B. Zamoldchikov, Nucl. Phys. **B133**, 525 (1978).
4. E. K. Sklyanin, L. A. Takhtadzhyan, and L. D. Faddeev, Teor. Mat. Fiz. **40**, 194 (1979).
5. J. Scherk, Rev. Mod. Phys. **47**, 123 (1975).
6. P. Dirac, Lekstii po kvantovoĭ mekhanike (Lectures on Quantum Mechanics), Moskov, 1968 [New York, Belfer Graduate School of Science, Yeshiva University, 1964].
7. B. M. Barbashov and N. A. Chernikov, Preprint J. I. N. R. P2-7852, Dubna, 1974.

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