Structure of Saturn's rings

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The stability of Saturn's F ring with respect to growth of the azimuthally periodic structure discovered by Voyager 1 is analyzed.

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In its recent flyby of Saturn the Voyager 1 spacecraft detected an azimuthally periodic structure in the F ring, which is the farthest from the planet. The appearance of the structure suggests two braided hoses.[1] Could such a structure form solely as a result of the gravitational forces of the planet and the ring material?

The motion of cold material in the plane of the ring is described by

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$$\frac{d\mathbf{v}}{dt} = -\nabla(\phi_o + \phi); \quad \frac{\partial\sigma}{\partial t} + \operatorname{div}(\sigma\mathbf{v}) = 0; \tag{1}$$

$$\phi_o = -\gamma M_s / r; \quad \Delta\phi = 4\pi\gamma\sigma\delta(z), \tag{2}$$

where σ is the surface mass density, $\phi_0 + \phi$ is the gravitational potential, M_S is the mass of Saturn, and r is the distance from the center of Saturn. If self-gravitation is ignored, the ring material will revolve at an angular velocity $\Omega = \sqrt{\gamma M_S / r^3}$.

We denote by R the radius of the F ring. Assuming that this ring is thin, we set $r = R + x$, and we transform to a coordinate system which is rotating at the angular velocity $\Omega_0 = \sqrt{\gamma M_S / R^3}$. In this new coordinate system, Eqs. (1) for the case of steady-state flow become

$$(\mathbf{v}\nabla)\mathbf{v} - 2[\mathbf{v}\vec{\Omega}_o] = 2\Omega_o \nabla F; \quad \operatorname{div}(\sigma\mathbf{v}) = 0, \tag{3}$$

where $F = 3/4\Omega_0 x^2 - \phi/2\Omega_0$.

Assuming that the period of the structure satisfies $\lambda \ll 2\pi R$, we ignore the curvature of the ring in (3) and replace the azimuthal angle θ by the coordinate $y = \theta R$. On the other hand, we assume that λ is much larger than the ring width Δ, and we correspondingly ignore the first nonlinear term on the left side of the x component of Eq. (3) in comparison with the second term. In this approximation we find

$$v_y = -\frac{\partial E}{\partial x}. \tag{4}$$

For a thin ring, $\Delta \ll \lambda$, the potential ϕ within the ring may be assumed independent of x, so that v_y is approximately equal to its unperturbed value: $v_y \simeq -(3/2)\Omega_0 x$. Substituting this expression in the y component of the first equation in (3), we find

$$v_y = 4\frac{\partial E}{\partial y} = -\frac{2}{\Omega_o}\frac{\partial \phi}{\partial y}. \tag{5}$$

Here ϕ must be understood as some variable ϕ which depends on the y part of the potential ϕ. According to (4) and (5), we have $v_y \simeq v_y(x)$ and $V_x \simeq v_x(y)$, so that the flow is incompressible in this approximation: $\operatorname{div} v = 0$. Thus we have $\sigma = \sigma(\Gamma)$, where the stream function Γ is defined by

$$v_x = \frac{\partial\Gamma}{\partial y}; \quad v_y = -\frac{\partial\Gamma}{\partial x}; \quad \Gamma = \frac{3}{4}\Omega_o x^2 - 2\tilde{\phi}/\Omega_o. \tag{6}$$

Here the potential ϕ is determined by Eq. (2), or

$$\Delta\tilde{\phi} = 4\pi\gamma\delta(z)\,[\sigma(\Gamma) - \langle\sigma(\Gamma)\rangle], \tag{7}$$

where $\langle\sigma\rangle$ is the value of σ averaged over the azimuthal direction.

The bifurcation point, i.e., the point at which an azimuthally homogeneous ring becomes unstable, is determined from the condition under which the linearized version of Eq. (7) has an eigenfunction solution. We assume that for a ring consisting of two "hoses" the unperturbed density $\sigma(x)$ can be approximated by

$$\sigma(x) = \sigma_0 \left(\frac{x}{\Delta}\right)^2 \exp(-x^2/\Delta^2), \qquad (8)$$

where $\sigma_0 = M_F/\pi^{3/2}\Delta R$, and M_F is the mass of the ring.

We have $\sigma = \sigma(\Gamma)$ for a periodically perturbed ring, so that x^2 in Eq. (8) must be understood as $4\Gamma/3\Omega_0$. Substituting in (7) the expression for Γ from (6) and linearizing the result, we find

$$\Delta\tilde{\phi} = -4\pi\gamma\delta(z)\sigma_0 \frac{8\tilde{\phi}}{3\Delta^2\Omega_0^2}\left(1 - \frac{x^2}{\Delta^2}\right)\exp\left(-\frac{x^2}{\Delta^2}\right). \qquad (9)$$

Thus we find the bifurcation condition for a thin ring, with logarithmic accuracy, to be

$$M_F/M_S = A\left(\frac{\Delta}{R}\right)^2, \qquad (10)$$

where $A = (3/32)\ln(\lambda/\Delta)$. The value of A is $\geqslant 10^{-2}$, and with the value $\Delta = 30$–50 km found from optical measurements[2] we find $M_F/M_S \sim 10^{-9}$ from (10) (with $R \approx 10^5$ km). Since the mass of the entire ring does not exceed $10^{-6} M_S$, the value $M_F/M_S \sim 10^{-9}$ seems possible.

When bifurcation point (10) is crossed, a periodic structure corresponding to successive decreases and increases in the distance between the hoses should begin to develop in the ring. We see from the structure of Eq. (7) that the instability and the appearance of the periodic potential ϕ result from a decrease in v_y and an increase in the density σ near those values of x which are points of the separatrix of the streamlines, $\Gamma = \text{const}$. As material is captured into "islands" with $\phi > 0$, the instability should reach saturation at a certain amplitude. If the visible emission is at a maximum near the separatrix, the picture which will be seen will be one of two braided hoses. It thus appears that we should not rule out the possibility that the visible structure of the F ring may be explainable by gravitational forces alone.

1. C. S. Suttun, New Scientist **88**, 491 (1980).
2. M. M. Waldrop, Science **210**, 1107 (1980).

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