Analytic continuation of the integral equations of the theory of scattering by an unphysical energy sheet

Yu. V. Orlov

Institute of Nuclear Physics, M. V. Lomonosov Moscow State University

(Submitted 9 February 1981)

Pis'ma Zh. Eksp. Teor. Fiz. **33**, No. 7, 380–384 (5 April 1981)

A simple rule is given for the Lippman-Schwinger equation for the analytic continuation of the integral equations with a Cauchy kernel on an unphysical energy sheet E ($\mathrm{Im}\sqrt{E}<0$). *The result obtained in this paper makes it possible to determine the virtual and resonance poles of the S matrix, and also the vertex decay (fusion) functions of the corresponding states by using the standard methods of calculating the bound states.*

PACS numbers: 03.80. + r

One of the important problems of the theory of scattering and nuclear reactions is the calculation of the resonant and virtual (antibound) states, which correspond to the poles of the S matrix on an unphysical energy sheet E of Riemann space (I_-, Im $\sqrt{E}<0$). Recently, interest in this problem has increased dramatically because of the large expansion of the range of physical problems in which resonances are the principal subject of study. It is sufficient to point to the study of dibaryon resonances,[1–3] resonances in $N\Delta$,[4–6] $N\bar{N}$,[7–8] and $2N\bar{N}$[9,10] systems, as well as resonances in nuclear reactions,[11] which are still of pressing interest. Unfortunately, the existing methods of finding the poles of the S matrix in I_- are cumbersome from a computational point of view and, as a rule, do not guarantee the correctness of the results. The latter is due to the need for an analytic continuation in I_- of the various physical quantities, which are given numerically for a number of points of the physical sheet E (I_+, Im $\sqrt{E}>0$). For example, the method of analytic continuation with respect to the coupling constant λ, which has obvious advantages over other approximate methods,[11] cannot be used effectively for finding virtual states and resonances far from the threshold. To accomplish this, it is necessary to calculate first the energies of the bound states for a large number of points in λ. An expansion of the effective-radius theory[12] is usually used in the near-threshold region. Resonances close to the physical region are found by calculating the Regge trajectories.[13] The algebraic relationship between the values of the t matrix at the upper (t_+) and lower (t_-) shores of the right-hand cut, which can be obtained from the Low equation and the unitarity of the S matrix, is known.[14] However, to find the poles of the S matrix by means of the analytic continuation of this relationship, we need to know the t matrix on the energy surface in the entire E region in I_+, corresponding to the E region in I_- where the pole is to be found.

In this paper we propose the most direct method of finding the poles of the S matrix in I_-, which consists of a rigorous analytic continuation of the integral equations of scattering theory, as a result of which the problem becomes completely ana-

 © 1981 American Institute of Physics

logous to the calculation of the energies of bound states. We shall illustrate this method by using the Lippman-Schwinger (LS) equation for the t matrix outside the energy surface with a potential $V(r)$, which is written in the form (z in I_+)

$$t_l(q, q', z) = V_l(q, q') + 4\pi \int_0^\infty \frac{V_l(q, k) t_l(k, q', z)}{z - k^2/2\mu} k^2 dk, \tag{1}$$

where

$$V_l(q, q') = (2\pi^2)^{-1} \int_0^\infty j_l(qr) V(r) j_l(q'r) r^2 dr, \tag{2}$$

$j_l(x)$ is a spherical Bessel function, μ is the reduced mass, q and q' are the initial and final momenta, and l is the orbital momentum which is omitted below to shorten the notation. For real z-E we have

$$t_\pm(q, q', E) = V(q, q') + 4\pi \int_0^\infty \frac{V(q, k) t_\pm(k, q', E)}{E - k^2/2\mu \pm i\epsilon} k^2 dk, \quad (\epsilon \to 0+) \tag{3}$$

where t_+ and t_- are the values of the t matrix at the upper and lower shores of the right-hand cut in the complex E plane. In order to find the LS equation in I_-, we must analytically continue the integral term in Eq. (1). To do this, we make use of the known method of finding the regular branch of a function that is expressed as a Cauchy integral, which originates from the initial branch of this multivalued function when the branch point (end of the integration) is bypassed along a closed contour.[15] The form of the integration contour is unimportant. For our problem we write the principal branch in the form

$$\Phi(z) = \frac{1}{2\pi i} \int_0^\infty \frac{\phi(\zeta) d\zeta}{\zeta - z}, \quad z \text{ in } I_+ \tag{4}$$

where ϕ satisfies the Hölder condition everywhere along the integration line. Using the well-known Sokhotskiĭ formulas

$$\frac{1}{\zeta - z \mp i\epsilon} = \frac{P}{\zeta - z} \pm i\pi\delta(\zeta - z), \quad \epsilon \to 0+, \tag{5}$$

where P denotes the principal Cauchy integral, we can see that the analytic continuation of $\Phi(z)$ from region I_+ to region I_- is the function

$$\Phi_-(z) = \Phi(z) - \phi(z). \tag{6}$$

Analytic continuation by means of Eq. (6) is valid for the z region in which the ϕ function is holomorphic. In order to use Eq. (6) for the analytic continuation of the integral in Eq. (1), we must remove the dependence of the t matrix on z outside the integral sign. To accomplish this, we expand $t(k, q', z)$ in a series in powers of $\sqrt{z}$ within a small neighborhood of the point $z = 0$. The specific form of the series is unimportant (it can be a Taylor, Laurent, or Mittag-Leffler series). To be specific, we write a Taylor series in the region where the t function is analytic with respect to k:

$$t(k, q', z) = \sum_{n=0}^{\infty} C_n(k, q')(\sqrt{z})^n. \tag{7}$$

Using Eq. (7) we can write the integral in the LS equation (1) in the region of uniform convergence of the series (7) in the form

$$\int_0^{\infty} \frac{V(q, k)\, t(k, q', z)}{z - k^2/2\mu} k^2\, dk = \sum_{n=0}^{\infty} (\sqrt{z})^n \int_0^{\infty} \frac{V(q, k)\, C_n(k, q')}{z - k^2/2\mu} k^2 dk. \tag{8}$$

After replacement of the variable ($\zeta = k^2/2\mu$), we obtain integrals of the form (4) as the coefficients of the series (8). By analytically continuing each term of the series (8) with the aid of Eq. (6) and summing the result in accordance with Eq. (7), we obtain an equation of the form (z in I_-)

$$t_-(q, q', z) = V(q, q') + 4\pi \int_0^{\infty} \frac{V(q, k)\, t_-(k, q', z)}{z - k^2/2\mu} k^2 dk + i8\pi^2\mu\sqrt{2\mu z}\; V$$

$$\times (q, \sqrt{2\mu z})\, t_-(\sqrt{2\mu z}, q', z), \tag{9}$$

which is the sought-for analytic continuation of the LS equation to the unphysical sheet. It is easy to prove the validity of the derivation of Eq. (9) by examining any potential that makes it possible to solve Eq. (1) [and consequently Eq. (9)] in analytic form. It is easy to show (for example, by using the approximation of the potential by the Bateman method with any number of cuts) that for a potential written in the form of an arbitrary sum of separable terms the solution of Eq. (9) coincides with the analytic continuation of the solution of Eq. (1) using Eq. (6). The analytic continuation of the solution of the LS equation (1), which was found by the Bateman method, was used[16] to determine the virtual state of the neutron-proton pair in the 1_{S_0} state. A more suitable form of the LS equation on the unphysical sheet can be obtained from Eq. (9) if $t_-(\sqrt{2\mu z}, q', z)$ is expressed using this equation. In this case the LS equation in I_- acquires the form of the usual Fredholm integral equation

$$t_-(q, q', p^2/2\mu) = V(q, q') + 8\pi\mu \int_0^{\infty} \frac{V(q, k)\, t_-(k, q', p^2/2\mu)}{p^2 - k^2} k^2 dk$$

$$+ F(q, p)\left[V(p, q') + 8\pi\mu \int_0^{\infty} \frac{V(p, k)\, t_-(k, q', p^2/2\mu)}{p^2 - k^2} k^2\, dk \right]. \tag{10}$$

where

$$F(q, p) = \frac{i8\pi^2\mu p V(q, p)}{1 - i8\pi^2\mu p V(p, p)}, \tag{11}$$

$p = \sqrt{2\mu z}$ is the arithmetic value of the root. It is easy to see from Eq. (9) for $t_-(p, q', p^2/2\mu)$ that the zero of the denominator in Eq. (1) is not a pole of the t matrix. By reducing Eq. (10) to an algebraic system of equations by means of some quadrature, we can find all the poles of t as a function of z in I_-, without solving Eq. (10), from the condition that the determinant of the corresponding system of algebraic equations is equal to zero in exactly the same way as the energies of the bound states are determined. We did not use the unitariy condition of the S matrix in the derivation of Eqs. (9) and (10); therefore, they can also be used formally for a complex potential (it must be remembered, however, that the very nature of the discontinuity of the scattering amplitude at the right-hand cut is closely related to the unitarity relation which is used, for example, in deriving the dispersion relations). The equation for the vertex function of the decay of the resonant or virtual state $g(q)$ follows immediately from Eq. (10). Near the corresponding pole at $z=z_0$ we can write

$$t_-(q, q', z) \approx \frac{g(q)\, g(q')}{z - z_0}. \tag{12}$$

Substituting Eq. (12) in Eq. (10) and dropping the terms that do not contain poles, we obtain the equation

$$g(q) = 4\pi \int_0^\infty \frac{[V(q, k) + F(q, p_0)\, V(p_0, k)]}{z_0 - k^2/2\mu}\, g(k)\, k^2\, dk. \tag{13}$$

Just as for a bound state, this equation is homogeneous; therefore, we must normalize $g(q)$ by using the equality (12)

$$g^2(q) = \lim_{z \to z_0} (z - z_0)\, t_-(q, q, z) = \tau(q, q, z_0), \tag{14}$$

where

$$\tau(q, q, z) = (z - z_0)\, t_-(q, q, z). \tag{15}$$

Thus, to determine the normalization of g it is sufficient to solve Eq. (10) at several points $z \neq z_0$ on the straight line passing through the point z_0, and then find $\tau(q, q, z_0)$ and, consequently, g^2 from Eq. (14) by means of interpolation.

In conclusion, we note that the method proposed in this paper for analytic continuation of the integral equations on the unphysical sheet is a multichannel problem, which makes it possible to derive the energies on any sheet of the Riemann surface. This applies to systems of LS equations, equations of the dispersion method, and to the Fadeev equations, which will be examined in a separate paper.

1. A. Yokosawa, In: Proc. of the Meeting on Exotic Resonances, Hiroshima, Japan, September 1–2, 1978.
2. I. P. Auer *et al.*, Nucl. Phys. A **335**, 193 (1980).
3. P. E. Argan, Phys. Rev. Lett. **46**, 96 (1981).
4. L. A. Kondratyuk and I. S. Shapiro, Yad. Fiz. **12**, 401 (1970) [Sov. J. Nucl. Phys. **12**, 220 (1971)].
5. V. B. Belyaev *et al.*, J. Phys. G **5**, 1057 (1979).
6. W. M. Kloet and R. R. Filbar, Nucl. Phys. A **338**, 281, 317 (1980); Phys. Rev. Lett. **45**, 970 (1980); W. M. Kloet *et al.*, Pseudoresonance behavior in nucleon-nucleon scattering, Preprint RU-80-238, 1980.
7. I. S. Shapiro, Usp. Fiz. Nauk **125**, 577 (1978) [Sov. Phys. Usp. **21**, 645 (1978)]; I. S. Shapiro,

Phys. Rep. C **35**, 129 (1978).
8. L. Montanet, Experim. review on the baryon-antibaryon interaction (submitted to the Vth Intern. Conf. on Meson Spectroscopy, Boston, April 29–30, 1977), Phys. Rep. **63**, 201 (1980).
9. O. D. Dal'karov *et al.*, Yad. Fiz. **17**, 1321 (1973) [Sov. J. Nucl. Phys. **17**, 688 (1973)].
10. Yu. V. Orlov and A. A. Tilyavov, Pis'ma Zh. Eksp. Teor. Fiz. **28**, 261 (1978) [JETP Lett. **28**, 241 (1978)]; Yad. Fiz. **30**, 497 (1979) [Sov. J. Nucl. Phys. **30**, 257 (1979)].
11. F. A. Gareev *et al.*, Preprint OIYaI, R4-12001, Dubna, 1978.
12. L. P. Kok, Phys. Rev. Lett. **45**, 427 (1980).
13. V. de. Alfaro and T. Regge, Potentsial'noe rasseyanie (Potential Scattering), Mir Press, Moscow, 1966.
14. J. E. Brown and A. S. Jackson, Nucleon-Nucleon Interactions (Russ. Transl., Atomizdat Press, Moscow, 1979).
15. F. D. Gakhov, Kraevye zadachi (Boundary-Value Problems), Fizmatgiz Press, Moscow, 1963.
16. V. B. Belyaev, B. F. Irgaziev, and Yu. V. Orlov, Yad. Fiz. **24**, 44 (1976) [Sov. J. Nucl. Phys. **24**, 22 (1976)].

Translated by Eugene R. Heath
Edited by S. J. Amoretty