Transfer processes in stochastic magnetic fields. Hydrodynamic approach

V. A. Rozhanskiĭ
M. I. Kalinin Leningrad Polytechnic Institute

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An estimate is obtained for the energy flow across magnetic surfaces that have been destroyed because of a "shivering" of the magnetic-field lines of force. It is shown that the electron thermal conductivity in tokamaks is given by the Ohkawa formula.

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Heat transfer across magnetic surfaces, which have been destroyed because of a "shivering" of the lines of force, has been investigated intensively recently (see, for example, Refs. 1–3). In these papers the electron thermal-conductivity coefficient $\chi_\perp$ has been related to the spectrum of fluctuations of the magnetic field $B_{\mathbf{k}}$. On the other hand, the fluctuation level of $B_{\mathbf{k}}$ is determined by the nonlinear saturation of the plasma oscillations, so that the problem must be solved self-consistently. We assume that the magnetic-field turbulence is caused by a buildup of the electron-temperature drift instability[4] and determine a relationship between the corresponding heat flux and the plasma parameters.

For simplicity, we consider the problem in two-dimensional geometry

$$\mathbf{B} = B_z \mathbf{e}_z + B_y(x)\,\mathbf{e}_y + \delta\mathbf{B};\quad B_y/B_z = \theta \ll 1,$$
$$\delta B_x = \sum_{\mathbf{k}} B_{\mathbf{k}x} = \sum_{\mathbf{k}} B^{(0)}_{\mathbf{k}x} \exp(ik_z z + ik_y y - i\omega t). \qquad (1)$$

The resonant magnetic surfaces correspond to

$$k_{\parallel}(x) = (k_z B_z + k_y B_y)/B \to 0. \qquad (2)$$

The system of hydrodynamic equations for the electrons at $\lambda_e \ll k_{\parallel}^{-1}$, where λ_e is the range of electrons, has the form

$$-i(\omega - k_y v_o)\, n_{\mathbf{k}} + i k_{\parallel} v_{e\parallel\mathbf{k}} - i k_y \phi_{\mathbf{k}} \frac{c}{B}\frac{dn_o}{dx} = 0$$

$$i(\omega - k_y v_o)\, m_e v_{e\parallel\mathbf{k}} - i k_{\parallel} T_{eo} n_{\mathbf{k}} - 1{,}71\, i k_{\parallel} n_o T_{e\mathbf{k}} + \qquad (3)$$

$$i k_{\parallel} e n_o \phi_{\mathbf{k}} - 0{,}51\, m_e \nu_{ei} n_o v_{e\parallel\mathbf{k}} - ,$$

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$$\frac{B_{\mathbf{k}x}}{B}\left(T_{eo}\frac{dn_o}{dx}+1{,}71\,n_o\frac{dT_{eo}}{dx}-en_o\frac{d\phi_o}{dx}\right), \qquad (4)$$

$$-{}^3\!/_2\,i(\omega-k_y v_o)T_{e\mathbf{k}}-{}^3\!/_2\,ik_y\frac{c}{B}\phi_{\mathbf{k}}\frac{dT_{eo}}{dx}+1{,}71\,ik_{\|}T_{eo}v_{e\|\mathbf{k}}+$$

$${}^3\!/_2\,k_{\|}^2\chi_{e\|}T_{e\mathbf{k}}-{}^3\!/_2\,ik_{\|}\chi_{e\|}\frac{dT_{eo}}{dx}\frac{B_{\mathbf{k}x}}{B}=0, \qquad (5)$$

where $v_0=c/B\cdot d\phi_0/dx$ and ϕx is the unperturbed electric field. For oscillations with a smaller transverse component of the wavelength than the ion Larmor radius the ions have a Boltzmann distribution

$$\phi_k=-\frac{T_{io}}{e}\frac{n_{\mathbf{k}}}{n_o}. \qquad (6)$$

The longitudinal electron current is related to the magnetic-field perturbation

$$\frac{4\pi}{c}en_o v_{e\|\mathbf{k}}=ik_y B_{\mathbf{k}x}. \qquad (7)$$

Ignoring the damping (the terms $3/2\chi_{e\|}k_{\|}^2T_{e\mathbf{k}}$ and $0.51\,m_e\nu_{ei}n_0v_{e\|\mathbf{k}}$) in Eqs. (3)-(7) and the terms containing $B_{\mathbf{k}x}/B$, we find that $\omega^3=-1.71\,k_yu_{Te}k_{\|}^2Ti_0/m_e$ at $\eta=d\ln T_{e0}/d\ln n_0\gg1$, where $u_{Te}=-(c\,dT_{e0}/dx)/eB$. For the estimates below we shall assume that $\lambda_e\sim k_{\|}^{-1}$. Damping is important at $k_{\|}v_{Te}\sim\omega_d=k_yu_{Te}$ [where $v_{Te}==(2T_{e0}/m_e)^{1/2}$]. Thus, the oscillations are localized near the resonant magnetic surfaces with a characteristic localization region Δx (Ref. 4)

$$\Delta x\sim\frac{a}{\theta}\frac{k_{\|}}{k_y}\sim\frac{\rho_{ce}}{\theta}, \qquad (8)$$

where ρ_{ce} is the electron Larmor radius and a is the characteristic scale in the x direction. The terms containing $B_{\mathbf{k}x}/B$ can be figured for $k_y^{-1}<c/\omega_{pe}$ (ω_{pe} is the plasma frequency of electrons), i.e., the transverse component of the wavelength does not exceed the size of the skin layer. This condition makes it possible to ignore the eddy electric field $E_y\sim\omega B_{kx}/k_zc$. For the largest wavelengths with k_y^{-1} of the order of Δx we obtain from this the inequality

$$\beta=\frac{n_o(T_{eo}+T_{io})8\pi}{B^2}<\theta^2. \qquad (9)$$

This condition is usually satisfied in today's tokamaks. In this case the relation $\theta>(m_e/m_i)^{1/2}$ is also valid, so that the transverse component of the wavelength does not, in fact, exceed the ion Larmor radius.

We examine the following mechanism of instability saturation. If the number of modes m is large, then there are many modes in a region of width $\sim\Delta x$. The magnetic fields of these modes combine to form the total magnetic field δB_x of Eq. (1), which leads to a wandering of the line of force along x. If the line of force is deflected a distance exceeding Δx along the characteristic length $k_{\parallel}^{-1}$, then the modes are stabilized. From this we have the estimate

$$\sum_{\mathbf{k}} \left| \frac{B_{\mathbf{k}x}}{B} \right|^2 \sim (k_{\parallel} \Delta x)^2 \sim \left(\frac{k_{\parallel}}{k_y}\right)^2 \sim \left(\frac{\rho_{ce}}{a}\right)^2. \tag{10}$$

In this case the level of fluctuations $n_{\mathbf{k}}/n_0 \sim c^2 \theta(\omega_{pe}^2 \rho_{ce} a \sqrt{m})^{-1}$ is lower than the level $n_{\mathbf{k}}/n_0 \sim (ak_y)^{-1} \sim \rho_{ce}/a\theta$, at which nonlinear interaction of the modes due to the buildup of small-scale oscillations on the gradient of the large-scale oscillations becomes important. The criterion is the condition

$$\beta > \theta^2/\sqrt{m} \sim (\rho_{ce}/a\theta)^{1/2}\, \theta^2. \tag{11}$$

To estimate the electron heat flow, we use the linear equations (3)–(7). Allowance for the nonlinear interaction between the modes in terms of the total magnetic field, which reduces to zero the increment of the oscillations, does not change the order of magnitude of the heat flux caused by the heat conductivity along the magnetic field. The heat flux is given by the expression

$$q = -3/2\, n_0 \chi_{e\parallel} \left\langle \sum_{\mathbf{k},\mathbf{k}'} \left(ik_{\parallel} T_{e\mathbf{k}} \frac{B_{\mathbf{k}x}}{B} + \frac{dT_{eo}}{dx} \frac{B_{\mathbf{k}x} B_{\mathbf{k}'x}}{B^2} \right) \right\rangle . \tag{12}$$

Expressing $T_{e\mathbf{k}}$ in terms of $n_{\mathbf{k}}$ and $v_{e\parallel\mathbf{k}}$ by means of Eq. (5), $n_{\mathbf{k}}$ in terms of $v_{e\parallel\mathbf{k}}$ in accordance with Eqs. (3) and (6), using Maxwell's equation (7), and substituting the value of $T_{e\mathbf{k}}$ in Eq. (12), we obtain

$$q = -n_o \chi_{e\parallel} \left\langle \sum_{\mathbf{k},\mathbf{k}'} \left\{ \frac{i k_{\parallel}^2 k_y c B T_{eo} \left(1{,}71 - \dfrac{T_{io}}{T_{eo}} \dfrac{k_y u_{Te}}{\omega - k_y v_o}\right)}{4\pi e n_o \left[i(\omega - k_y v_o) - k_{\parallel}^2 \chi_{e\parallel}\right]} \frac{B_{\mathbf{k}x} B_{\mathbf{k}'x}}{B^2} + \frac{3}{2} \frac{dT_{eo}}{dx} \frac{B_{\mathbf{k}x} B_{\mathbf{k}'x}}{B^2} \right\} \right\rangle . \tag{13}$$

Turning to the collisionless case, i.e., replacing λ_e by $k_{\parallel}^{-1}$, and assuming that $\omega \sim \omega_d = k_y u_{Te} \sim k_{\parallel} v_{Te}$ and $k_y^{-1} \sim \Delta x$, we obtain the estimate

$$\chi_{\perp} \sim \sum_{\mathbf{k}} \left(\frac{k_y c B a}{4\pi e n_o} + \frac{v_{Te}}{k_{\parallel}} \right) \left| \frac{B_{\mathbf{k}x}}{B} \right|^2 \tag{14}$$

for the effective transverse electron heat conductivity $\chi_{\perp}$. The second term in this expression corresponds to the result of Ref. 1 after replacing $\delta(k_{\parallel})$ by $k_{\parallel}^{-1}$ ($k_{\parallel} \lesssim k_y u_{Te}/v_{Te}$). It is easy to see, however, that for the condition (9) the first term, which exceeds the second term by a factor of θ^2/β, is the main term. Substituting the estimate (10) in Eq. (14), we obtain the relationship between $\chi_\perp$ and the plasma parameters

$$\chi_\perp \sim \frac{c^2}{\omega_{pe}^2} \frac{v_{Te}\,\theta}{a} . \tag{15}$$

This expression was proposed by Ohkawa[5] to explain the empirical dependence of the energy lifetime in tokamaks on the plasma density.

Thus, for the condition $\beta < \theta^2$ the characteristic scale of the oscillations is determined by the quantity $\sim\rho_{ce}/\theta$, the characteristic frequency is $\omega \sim \omega_d$, the characteristic wavelength along the magnetic field (the correlation length of the fluctuations δB_x) is $\lambda_\parallel \sim a/\theta$, and the electron heat-conductivity coefficient is given by the formula (15).

In the opposite case $\beta > \theta^2$ ($\rho_{ce}/\theta > c/\omega_{pe}$) the characteristic scale of the oscillations is determined by the size of the collisionless skin layer c/ω_{pe}. In this case $\omega \sim \omega_d$, $\lambda_\parallel \sim (a/\theta)(\theta/\sqrt{\beta}) < a/\theta$, and the saturation level, as before, is determined in accordance with Eq. (10). The first and second terms in Eq. (14) are of the same order of magnitude, and the value of $\chi_\perp$ is given by the estimate:

$$\chi_\perp \sim \left(\frac{c}{\omega_{pe}}\right) \frac{c\,T_{eo}}{eBa} . \tag{16}$$

The proposed model differs significantly from that analyzed in Refs. 6 and 7, where it was assumed that $k_y^{-1} \sim c/\omega_{pe}$, $k_\parallel^{-1} \sim a/\theta$, and $\omega \sim v_{Te}\theta/a$.

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