Evolution of small perturbations of isotropic cosmological models with one-loop quantum gravitational corrections

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Equations have been derived for small perturbations of homogeneous isotropic cosmological models, which take into account in the one-loop approximation the quantum gravitational effects produced as a result of interaction of the quantum fields of matter with the self-consistent gravitational field.

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The quantum gravitational effects can be taken into account in the one-loop approximation by adding to the right side of the Einstein equations the average energy-momentum tensor $\langle T_i^{\ k}\rangle$ of all the quantum fields, including the effect of fluctuations of the gravitational field. The obtained equations should be interpreted as the equations for the average values of the space-time metric. These equations apply if the quantum fluctuations of the metric are small compared with its average values. This occurs if the occupation numbers of all the considered modes of the gravitational field $n_k \gg 1$. This condition clearly is satisfied for homogeneous metrics if $|R_{iklm}R^{iklm}| \ll l_g^{-4}$, $l_g = \sqrt{G\hbar/c^3}$ (below we shall assume that $c = \hbar = 1$).

Suppose that the metric under consideration is given by

$$ds^2 = dt^2 - a^2(t)(\gamma_{\alpha\beta} + h_{\alpha\beta})\, dx^\alpha dx^\beta, \tag{1}$$

where α and $\beta = 1, 2, 3$, $\gamma_{\alpha\beta}$ is a three-dimensional constant-curvature metric equal to 1, 0, or -1 (these three cases are denoted by $\mathscr{H} = 1, 0, -1$, respectively), and $|h_\alpha^\beta| \ll 1$. To obtain equations for $a(t)$ and $h_{\alpha\beta}$ in a one-loop approximation, we must calculate $\langle T_i^{\ k}\rangle$ for the metric (1).

We assume that the quantum fields are free and examine the most interesting region $|R_{iklm}R^{iklm}| \gg m^4$, where the rest mass m of particles can be ignored in first approximation (it is understood that $ml_g \ll 1$). We also assume that all the quantum fields (or at least most of them) become conformally covariant as $m \to 0$. Thus, in zero order in h, $\langle T_i^k \rangle$ is comprised on the classical part (the contribution from the free particles) and the local terms associated with the conformal anomaly of the track (see, for example, Ref. 1).

$$< T > = - \frac{1}{2880\pi^2}\left[k_1 C_{iklm}C^{iklm} + k_2\left(R_{ik}R^{ik} - \frac{1}{3}R^2\right) + k_3 \Box R\right]. \tag{2}$$

The k_1, k_2, and k_3 constants depend on the field; for photons, for example, $k_1 = -13$,

$k_2 = 62$, and $k_3 = -18$. To ensure that the classical solutions of the Einstein equations are stable, we must assume that $k_3 < 0$. According to Ref. 2, we introduce the specifications $H^2 = 360\pi/Gk_2$ and $M^2 = -(360\pi/Gk_3)$. A nonlocal vacuum polarization appears in the first order in h; the production of real particles, which is proportional to h^2, is given by (Ref. 3)

$$\frac{1}{\sqrt{-g}} \frac{d\sqrt{-g}h}{dt} = \frac{\epsilon}{960\pi} C_{iklm}C^{iklm}, \tag{3}$$

where $\xi = \frac{1}{3}(2k_1 + k_2)$; $\xi = 1$ for neutral scalar particles, $\xi = 6$ for four-component fermions with spin $\frac{1}{2}$, and $\xi = 12$ for photons. First, we shall consider the special case $\mathscr{K} = 0$, $h_\alpha^\beta = h_\alpha \delta_\alpha^\beta$. The answer for the scalar field can be obtained from the results of Ref. 4 in the following way. The conformally covariant nonlocal part of $\langle T_i^k \rangle$ differs from that in Appendix I of Ref. 4 only in the factor a^{-4}. The local part of $\langle T_i^k \rangle$, which appears after the third subtraction, is contained in the integrals of the quantities $s^{(2)}$, $u^{(2)}$, $\tau^{(3)}$, $s^{(4)}$, and $u^{(4)}$ in Ref. 4. After lengthy but straightforward calculations using Eqs. (22) and (II.1) of Ref. 4, we find

$$\epsilon \equiv \langle T_0^0 \rangle = 0(h^2),$$

$$p_\beta \equiv -\langle T_\beta^\beta \rangle = \frac{1}{960\pi^2 a^4}\left\{\frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-i\omega\eta}\omega^4\left(\ln\frac{m^2a^2}{\omega^2} + i\pi\,\mathrm{sgn}\,\omega\right)h_\beta(\omega)\,d\omega + 2\frac{a'}{a}h_\beta''' + \frac{2}{3}h_\beta''\left[2\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2\right] + \frac{1}{3}h_\beta'\left[\left(\frac{a'}{a}\right)'' + 4\left(\frac{a'}{a}\right)'\frac{a'}{a}\right]\right\}, \tag{4}$$

where $h_\beta(\omega) = \int_{-\infty}^{\infty} h_\beta(\eta)e^{i\omega\eta}d\eta$, $d\eta = dt/a$, and the prime denotes differentiation with respect to η. The expression for $\langle in|T_i^k|out\rangle/\langle in|out\rangle$ differs from (4) only in that $\mathrm{sgn}\,\omega$ is missing in it in the integrand. The result of the calculation of (4) coincides with that obtained in Ref. 5 to within an accuracy of a factor of 2.

If the nonlocal part of $\langle T_i^k \rangle$ and the conformal anomaly are known, we can easily determine the expression for the quantum part of $\langle T_i^k \rangle$ in the first order in h for the general form of the weakly nonconformal metric

$$ds^2 = a^2(x^i)\left[dx_0^2 - \sum_{\beta=1}^{3} dx_\beta^2 + h_{kl}(x^m)\,dx^k dx^l\right], \tag{5}$$

where $|h_k^l| \ll 1$. The metric (1) is a special case of (5). We have

$$8\pi G \langle T_i^k \rangle = M^{-2}A_i^k + H^{-2}B_i^k + \frac{G\xi}{120\pi a^4}C_i^{k},$$

$$A_i^k = \frac{1}{6}\left(2\delta_i^k R_{;l}^{;l} - 2R_{;i}^{;k} + 2RR_i^k - \frac{1}{2}\delta_i^k R^2\right), \tag{6}$$

$$B_i^k = R_i^l R_l^k - \frac{2}{3} R R_i^k - \frac{1}{2}\delta_i^k R_l^m R_m^l + \frac{1}{4}\delta_i^k R^2 + 2C_{il}{}^{km}R_m^l ,$$

$$C_i^k = -\frac{1}{(2\pi)^4}\int d^4 q e^{-iqx} H_i^k(q)\left[\ln\frac{|q^2|}{m^2 a^2} - i\pi\theta(q^2)\,\mathrm{sgn}\, q_0\right] +$$

$$+ 4(C_{il}{}^{km}{}_{;m}\sigma^{,l} + C_{il}{}^{km;l}\sigma_{,m} + C_{il}{}^{km}\sigma_{;m}^{;l}),$$

$$H_i^k = 2C_{il}{}^{km;l}{}_{;m} + C_{il}{}^{km}R_m^l , \; C_i^i = H_i^i = 0, \quad \sigma = \ln a, \; qx \equiv q_i x^i .$$

Here $A_i{}^k$ and $B_i{}^k$ are determined from the metric (5) and $C_i{}^k$ and $H_i{}^k$ are determined from the metric enclosed in the square brackets in (5) (i.e., without the use of the conformal factor a). It is sufficient to omit sgn q_0 in $C_i{}^k$ for the transition to $\langle \mathrm{in}|T_i{}^k|\mathrm{out}\rangle/\langle \mathrm{in}|\mathrm{out}\rangle$. If $h_i{}^k$ depends only on η, and $k_1 = k_2 = k_3 = 1$, then Eq. (6) will change to Eq. (4). The term with $C_i{}^k$ in (6) must be retained only if the conditions $|C_{iklm}C^{iklm}| \gg m^4$ and $|q^2|/m^2a^2 \gg 1$ are satisfied; otherwise, it should be dropped. Ther terms with $A_i{}^k$ and $B_i{}^k$ are valid if the much weaker condition $|R_{iklm}R^{iklm}| \gg m^4$ is satisfied. The $A_i{}^k$ tensor satisfies the covariant conservation law exactly: $A^k_{i;k} = 0$, and the $B_i{}^k$ and $C_i{}^k$ tensors satisfy the covariant conservation law with an accuracy to values proportional to h^2; specifically, $B^k_{i;k} = 2C_{iklm}C^{nklm}{}_{;n}$. The local terms in $C_i{}^k$ are obtained by varying the action $\int d^4x\sqrt{-g}C_{iklm}C^{iklm}\sigma$.

The result for $\langle T_i{}^k\rangle$ given in Ref. 6 differs from (6) by the absence of local terms in $C_i{}^k$; a conformally invariant cutoff pulse λ, instead of the mass m, in this case is under the logarithm sign. Such an approach, however, is intrinsically contradictory, if we ignore the local terms in $C_i{}^k$ and retain $A_i{}^k$ and $B_i{}^k$, since there are generally no conformal anomalies in the theory with a conformally invariant cutoff parameter.[7]

After substituting $\langle T_i{}^k\rangle$ on the right side of Einstein euqations for the metric (1), we obtain the equations for small perturbations. These equations, which are a generalization of the classical Lifshitz equations,[8] become the latter M and $H\to\infty$ and $\xi\to 0$. Specifically, at $\mathscr{K}=0$ for tensor perturbations (gravitation waves) $h_\alpha^\beta = g_k(\eta)e_\alpha^\beta\exp(i\mathbf{kr})$, where e_α^β is the polarization tensor, in the absence of free particles we have $(k\equiv|\mathbf{k}|)$

$(k \equiv |\mathbf{k}|)$:

$$g_k''\left(1 - \frac{R}{3M^2} + \frac{R - 2R_0^\circ}{3H^2}\right) + g_k'\left[2\frac{a'}{a}\left(1 - \frac{R}{3M^2} + \frac{2R_0^\circ}{3H^2}\right) - \frac{R'}{3M^2}\right]$$

$$+ k^2 g_k\left(1 - \frac{R}{3M^2} + \frac{6R_0^\circ - R}{3H^2}\right) = \frac{G\xi}{60\pi a^2}\left[-\frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-i\omega\eta}(\omega^2 - k^2)^2\right.$$

$$\left.\times L(\omega)\, g_k(\omega)\, d\omega + 2\frac{a'}{a}\left(g_k''' + k^2 g_k'\right) + \left(\frac{a'}{a}\right)'(g_k'' - k^2 g_k)\right], \qquad (7)$$

where

$$g_k(\omega) = \int_{-\infty}^{\infty} g_k(\eta)e^{i\omega\eta}d\eta, \quad L(\omega) = \ln\frac{|\omega^2 - k^2|}{m^2a^2(\eta)} - i\pi\theta(\omega^2 - k^2)\,\mathrm{sgn}\,\omega$$

and the quantities R and R_0^0 are calculated from the unperturbed metric (for $h_\alpha^\beta = 0$).

An important, heretofore unnoticed conclusion follows from Eq. (6): In the absence of a classical material the scalar perturbations against the background of a de Sitter quantum stage[2] are conformally plane ($\delta C_{iklm} = 0$). The gravitation waves against this background are described exactly by the Lifshitz equation[8]; moreover, all the invariants comprised of the Weyl tensor vanish. In accordance with the above remark, the term with C_i^k in Eq. (6) must be dropped. It also follows from Eq. (7) that the nonsingular isotropic model of the universe constructed in Ref. 2 can exist only when $M < 2H$. Otherwise, it becomes strongly anisotropic before reaching the Friedmann stage at the moment when

$$1 - \frac{R}{3M^2} + \frac{R - 2R_0^0}{3H^2} = 0.$$

Equation (7) makes it possible to calculate the time τ_g of the decay of scalarons into gravitons. Substituting in (7) the evolution law at the scalaron stage[2]

$$a(t) = a_1 t^{2/3}\left[1 + \frac{2}{3Mt}\sin M(t - t_1)\right].$$

we find $\tau_g = (3/GM^3)(k_3/k_1)^2$. The decay of scalarons into gravitons is forbidden in theories in which $k_1 = 0$ (Ref. 9) (so that the conformal anomaly is missing when $R_{ik} = 0$).

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