Solitonlike solutions for a Higgs scalar field

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The interaction of two solitons is investigated in a two-dimensional classical scalar-field theory described by the Landau-Ginzburg equation. It is shown that a new solution (a double soliton) is formed and that the solution is self-localized, weakly oscillating, and weakly damped.

Usually the stability of a soliton is attributed in relativistic theory to the fact that the soliton solution differs topologically from the vacuum solution. In particular, the equation of motion

$$\phi_{tt} - \phi_{xx} - m^2\phi + \lambda\phi^3 = 0 \tag{1}$$

has a soliton solution

$$\phi(x, t) = \frac{m}{\sqrt{\lambda}}\,\mathrm{th}\,\frac{m(x - vt)}{\sqrt{2(1 - v^2)}}. \tag{2}$$

This solution has different boundary conditions at infinity with respect to x and therefore cannot go over into small oscillations about one of the vacuum values $\phi_\pm = \pm\sqrt{m^2/\lambda}$. The existence of a stable double soliton for the sine-Gordon equation is usually attributed to a fortuitous circumstance, namely, the complete integrability of this equation.

Let us examine the interaction of two soliton solutions of type (2). As the initial condition we take the function

$$\phi_0(x, 0) = \frac{m}{\sqrt{\lambda}}\left(\mathrm{th}\,\frac{m(x + x_0)}{\sqrt{2(1-v^2)}} + \mathrm{th}\,\frac{m(-x + x_0)}{\sqrt{2(1 - v^2)}} - 1\right). \tag{3}$$

It is shown in Fig. 1 (curve 1). The boundary conditions for this function coincide with the vacuum value $\phi_- = -\sqrt{m^2/\lambda}$. One could therefore hope that "annihilation" would take place upon collision of the solitons, i.e., that our solution would go over into a large number of small oscillations about the value of ϕ_-. The initial velocity at which the solitons approach each other was assumed equal to 0.1 (the velocity of light is $c = 1$).

Figure 1 shows the successive time evolution of the solution of Eq. (1) with initial condition (3) (numerical calculation). It is seen that the character of the motion is different: after the walls come closer together, large oscillations of patently linear origin take place in the vicinity of the point $x = 0$. Figure 2 shows the time dependence of the field ϕ at the point $x = 0$.

The observed coalescence of two solitons can be explained in the following manner: Assume that the exact solution at an arbitrary instant of time is of the form $\phi(x, t) = \phi_0(x, t) + f(x, t)$, with $f(x, t) \ll \phi_0(x, t)$ and with $\phi_0(x, t)$ representing a superposition of the two solitons and the vacuum ϕ_-. In the linear approximation, the equation for the function f is

$$f_{tt} - f_{xx} - m^2 f + 3\lambda\phi_0^2 f = Q(x, t), \tag{4}$$

where $Q(x, t)$ is the source of the field and f depends only on the function $\phi_0(x, t)$. The quantity $Q(x, t)$ has a maximum at $x = 0$, and the expression for it at this point takes the simple form

$$Q(0, t) = -6m^2\sqrt{\frac{m^2}{\lambda}}\,y(1 - y)^2, \tag{5}$$

where $y = \tanh[(x_0 - vt)m/\sqrt{2(1 - v^2)}]$. The radiation vanishes when the distance between the solitons is large ($y \approx 1$), and also when it is equal to zero ($y = 0$)—this arrangement of the solitons is equivalent to the exact vacuum solution ϕ_-. The value of $Q(0, t)$ increases when the solitons come closer together, and therefore the amplitude f of the waves increases with time. This is observed also in the numerical solutions (these oscillations are small and are therefore not shown in Fig. 1).

Let us examine now the behavior of the approximate solution $\phi_0(x, t)$. Neglecting the radiation, the Hamiltonian $H(\phi_0) = T(\phi_0) + V(\phi_0)$ should be conserved, and the condition for its conservation determines the equation of motion of the two solitons relative to each other. The behavior of the potential energy

$$V(\tilde{x}_0) = \frac{1}{2}\int\left\{\phi_{0x}^2 - m^2\phi_0^2 + \frac{\lambda}{2}\phi_0^4\right\}dx$$

as a function of the distance x_0 between the solitons is shown in Fig. 3. The behavior of $V(\tilde{x}_0)$ can be qualitatively explained by locating the solution $\phi_0(\tilde{x}_0)$ in the relief of the potential energy of the field. At large distances $\tilde{x}_0$, the potential energy is constant and equal to the sum of the masses of the two solitons, i.e., to $(4\sqrt{2}/3)(m^3/\lambda)$. When the solitons touch, the energy be-

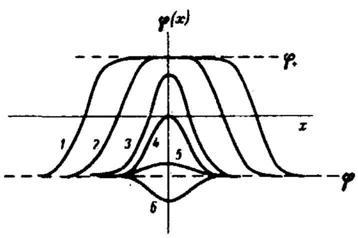

FIG. 1. Dependence of the solution on the time. The numbers indicate the time-ordered solutions $\phi(x, t_i)$ up to plot 6 inclusive, after which the plots 5, 4, 3, 4, 5, 6, etc. repeat again.

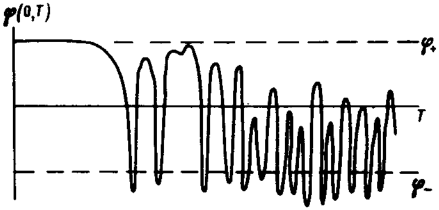

FIG. 2. Dependence of the oscillations of the field ϕ at the point $x=0$ on the time.

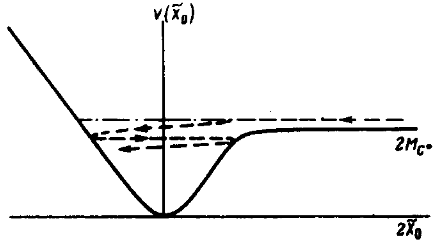

FIG. 3. Potential energy of two solitons separated by a distance $\tilde{x}_0$. The dashed line shows the soliton "capture" trajectory.

comes equal to zero, since it coincides with the vacuum energy. At negative $\tilde{x}_0$, the function $V(\tilde{x}_0)$ increases linearly with increasing $|\tilde{x}_0|$, since part of the solution becomes equal to $-3\sqrt{m^3/\lambda}$. For this value of the field, the potential energy exceeds the vacuum value.[1)]

It becomes clear from Fig. 3 that there can exist two different forms of soliton motion—infinite for energies exceeding the sum of the two soliton masses and finite in a well at lower energies.

Allowance for radiation modifies the picture. The energy $H(\phi_0)$ is no longer conserved, but decreases slowly. Therefore "capture" takes place when the solitons come close together, and the infinite motion becomes finite.

The period of the oscillations shown in Fig. 2 is equal to the period of the motion in the well and is of the order of m^{-1}. Since the vibrational motion of the solitons in the well takes place about the point $\tilde{x}_0=0$ at which the radiation is rigorously equal to zero, the radiation is strongly suppressed. Thus, the double soliton is a quasistable object.

With increasing initial soliton energy, the capture process ceases to be observed—the solitons repel each other and part of their energy is lost to radiation.

Analysis of the interaction of a larger number of solitons in such a potential scheme does not lead to the appearance of new many-soliton bound states.

It is clear that quantization of the soliton motion, in analogy with the quantization carried out in [1], leads to a discrete spectrum of heavy resonances with masses from zero to $2M_c$. These would be narrow resonances capable of decaying into one another with emission of light stable bosons of mass $\sqrt{2}m$.

The obtained new solution may turn out to be useful for the construction of a theory of Ψ bosons on the basis of coherent field states, as recently proposed by I. S. Shapiro.[2]

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[1)]The growth of the potential energy at negative $\tilde{x}_0$ is the cause of the soliton repulsion in this model. This fact was first noted by N. A. Voronov.

[1]V. E. Korepin, P. P. Kulish, and L. D. Feddeev, ZhETF Pis. Red. **21**, 302 (1975) [JETP Lett. **21**, 138 (1975)].
[2]I. S. Shapiro, ZhETF Pis. Red. **21**, 624 (1975) [JETP Lett. **21**, 293 (1975)].