Instability and nonlinear oscillations of solitons

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(Submitted July 10, 1975)
Pis'ma Zh. Eksp. Teor. Fiz. **22**, No. 7, 364–367 (5 October 1975)

The method of the inverse scattering problem is used to solve exactly the problem of soliton stability in media with dispersion relative to transverse perturbations. Solutions are obtained, which describe the nonlinear stage of the instability and (in the stable case) the nonlinear oscillations of the soliton.

PACS numbers: 03.65.Ge, 03.40.Kb

1. The question of soliton stability in media with weak dispersion relative to buildup of transverse oscillations is conveniently solved within the framework of the equation

$$\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial^2}{\partial x^2}\left(-\nu^2 u + \frac{1}{4} u_{xx} + \frac{3}{4} u^2\right) + \frac{3}{4}\beta^2 \frac{\partial^2 u}{\partial y^2}, \tag{1}$$

which was derived by Kadomtsev and Petviashvili,[1] who have shown that the soliton

$$u_0(x) = \frac{2\nu^2}{\mathrm{ch}^2 \nu x}, \tag{2}$$

is a stationary solution of (1) and is stable against transverse oscillations if $\beta^2 > 0$, and unstable if $\beta^2 < 0$. If $u = u_0 + \delta u$ and $\delta u \sim \exp[i\Omega t + ipy]$, then

$$\Omega^2 = \beta^2 \nu^2 p^2 + \ldots \ . \tag{3}$$

in a medium with dispersion length λ, the cases $\beta^2 \gtrless 0$ are realized if the dispersion law is of the form

$$\omega_k^2 = c^2 k^2 (1 \pm \lambda^2 k^2 + \ldots).$$

It was noted in[2] that the method of the inverse scattering problem is applicable to Eq. (1) (see also[3]). If we take a function $F(x, z, y, t)$ satisfying the two equations

$$\frac{\partial F}{\partial t} = \frac{\partial^3 F}{\partial x^3} + \frac{\partial^3 F}{\partial z^3} - \nu^2\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\right), \tag{4}$$

$$\beta \frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial z^2} = 0 \tag{5}$$

and solve for all values of x, y, and t the integral equation

$$F(x, z, y, t) + K(x, z, y, t) + \int_x^\infty K(x, s, y, t) F(s, z, y, t)\, ds = 0, \tag{6}$$

then the quantity

$$u(x, y, t) = 2\frac{d}{dx} K(x, x, y, t)$$

satisfies Eq. (1) even if u belongs to any matrix algebra. In particular, for the soliton (2) we have

$$F = F_0 = 2\nu e^{-\nu(x+z)}; \qquad K = K_0 = \frac{2\nu e^{-\nu(x+z)}}{1 + e^{-2\nu x}} .$$

The purpose of the present note is to obtain, by the inverse-problem method, an exact expression for $\Omega^2(p)$, and also to investigate the nonlinear soliton oscillations (at $\beta^2 > 0$) and the nonlinear stage of its instability (at $\beta^2 < 0$).

2. Let u, F, and K be matrices in the form

$$\begin{bmatrix} a_0 & a_1 \\ 0 & a_0 \end{bmatrix}.$$

Then u_1 satisfies the linearized equation

$$\frac{\partial^2 u_1}{\partial x \partial t} = \frac{\partial^2}{\partial x^2}\left(-\nu^2 u_1 + \frac{1}{4} u_{1xx} + \frac{3}{2} u_0 u_1\right) + \frac{3}{4}\beta^2 \frac{\partial^2}{\partial y^2} u_1.$$

Putting $F = \phi(y, t)\exp[-\eta x - \kappa z]$, we obtain from[6]

$$K_1(x, x; y, t) = \phi(y, t)e^{-(\eta+\kappa)x}\left(-1 + \frac{2\nu}{\nu+\kappa}\frac{1}{1+e^{2\nu x}}\right)\left(1 + \frac{1}{1+e^{2\nu x}}\right).$$

From the condition that $u_1(x,y,t)$ decrease as $x \to \pm\infty$, we obtain $\kappa = \nu$ and $\mathrm{Re}(\nu - \eta) > 0$. From the conditions for the compatibility of Eqs. (4) and (5) for F, we get

$$\phi(y, t) = \phi_0 e^{i\Omega t + ipy} \qquad \Omega^2 = \beta^2 p^2(\nu^2 - i\beta p). \tag{7}$$

The sign of β must be chosen from the condition $\mathrm{Im}\,\Omega \geq 0$ as $|p| \to \infty$. At $\beta^2 > 0$ formula (11) describes the spectrum of damped oscillations, and at $\beta^2 < 0$ it describes the soliton instability growth rate. As $|p| \to 0$ Eq. (7) leads to the result of Kadomtsev and Petviashvili [Eq. (3)]. In terms of physical variables, the soliton amplitude is described by the deviation $\delta c/c$ of its velocity from that of sound, and Eq. (1) is applicable if the soliton thickness is $lx \sim \lambda(c/\delta c)^{1/2} \ll ly$. In terms of these variables, the minimal transverse instability scale is $ly \sim \beta/\nu^2 \sim \lambda c/\delta c \gg lx$, which justifies the applicability of formula (7) at $\delta c/c \ll 1$.

3. Let us consider a broader class of exact solutions of (1). Let u be a scalar and let $\beta^2 < 0$. We choose the function F in the form

$$F = \Psi(x, y, t)\Psi^*(z, y, t).$$

From the conditions for the compatibility of (4) and (5) we have

$$\Psi(x, y, t) = \int a(\eta)e^{\eta(\nu^2-\eta^2)t + i(\eta^2-\nu^2)y - \eta x}d\eta\,. \tag{8}$$

Solving Eq. (6), we obtain the exact solution of (1) in the form

$$u(x, y, t) = -2\frac{d}{dx}\frac{|\Psi(x, y, t)|^2}{1 + \int_x^\infty |\Psi(x, y, t)|^2 dx}\,. \tag{9}$$

In particular, if

$$\Psi(x, y, t) = \int_0^\nu a(\eta)e^{\eta(\nu^2-\eta^2)t + i(\eta^2-\nu^2)y - \eta x}d\eta + \sqrt{2\nu}e^{-\eta x} \tag{10}$$

we obtain a solution that tends to the soliton (2) as $t \to \infty$ and describes the soliton instability. By investigating the case $a(\eta) = a\delta(\eta - \eta_0)$ with $\eta_0 < \nu$ we verify that as $t \to +\infty$ the solution (9) goes over into a soliton of smaller amplitude $2\eta_0^2$ and a vibrational "background" that decreases uniformly with x. Letting $\eta_0 \to 0$ we can cause the initial soliton to vanish completely.

The possibility of complete vanishing of the soliton as a result of instability development is due to the fact that at $\beta^2 < 0$ the soliton moves with subsonic velocity $\delta c < 0$ and can give up energy to the small material oscillations that overtake it. The intermediate picture of instability development depends on the concrete form of the function $a(\eta)$.

4. In the stable case $\beta^2 > 0$ it is also possible to construct for Eq. (1) an exact solution that depends on an arbitrary function. We put

$$F = \chi(x, y, t)e^{-\nu z}\,.$$

Then

$$u(x, y, t) = -2\frac{d}{dx}\frac{\chi(x, y, t)e^{-\nu x}}{1 + \int_x^\infty \chi(x, y, t)e^{-\nu x}dx}\,, \tag{11}$$

where we get from the condition of the compatibility of (4) and (5)

$$\chi(x, y, t) = \int_{-\infty}^{\infty} c(k)\exp[iky - \eta x + \eta(\eta^2 - \nu^2)t]dk; \quad \eta^2 = \nu^2 - ik.$$

If $c(k) = 2\nu\delta(k) + \tilde{c}(k)$, then the solution (11) describes damped nonlinear oscillations of the soliton. In the stable case $\beta^2 > 0$ the soliton is supersonic, and its oscillations are damped by the radiation of the sound that lags it.

The possibility of soliton vanishing through instability at $\beta^2 < 0$ causes the shock waves in media with positive dispersion $\omega_k'' > 0$ to have an essentially turbulent structure.

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