Inhomogeneous stationary states in the Heisenberg model

V. L. Pokrovskiĭ and S. B. Khokhlachev

L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences
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The exact wave functions of solitons and the corresponding eigenvalues of the Hamiltonian are obtained.

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In this paper we construct the exact wave functions and obtain the eigenvalues of the Hamiltonian

$$H = -\sum_{x,a} J_{\alpha\beta}\, s^{\alpha}_{x+a}\, s^{\beta}_{x}\ , \tag{1}$$

where $J_{\alpha\beta}$ is a symmetric and positive-definite matrix, s^{α}_{x} is the spin matrix, **x** numbers the lattice points, and the sum over **a** denotes summation over the nearest neighbors. The spin operators obey the usual commutation relations. We introduce a local rotation operator $R(\mathbf{x})$ acting on the spin matrices in accordance with the rule

$$R^{+} s^{a} R = R^{a\beta} s^{\beta}, \tag{2}$$

where $R^{\alpha\beta}$ is an orthogonal matrix. In formula we have left out, for brevity, the index **x**. The stationary state is sought in the form

$$|\Psi\rangle = \prod_{x} R(x)\,|\,0\rangle\ , \tag{3}$$

where $|0\rangle$ denotes the ground state in the Heisenberg model. We shall assume that in this state all the spins have a maximum possible projection on the "3" axis. In order for the state (3) to be stationary it suffices for the local rotation matrix to satisfy the system of difference equations

$$\sum_{a} J_{\alpha\beta}\ n^{\alpha}(x+a)\, e^{\beta}_{+}(x) = 0\ , \tag{4}$$

$$J_{\alpha\beta}\, e^{\alpha}_{+}(x+a)\, e^{\beta}_{+}(x) = 0. \tag{5}$$

The notation in (4) and (5) is the following:

$$n^{\alpha}(x) = R^{\alpha\beta}(x)\, e^{\beta}_{3} \tag{6}$$

$$e^{\alpha}_{+}(x) = R^{\alpha\beta}(x)\,(e^{\beta}_{1} + i\, e^{\beta}_{2}), \tag{7}$$

where $\mathbf{e}_i$ is a system of unit vectors directed along the principal axes of the tensor $J_{\alpha\beta}$ (we recall that $\mathbf{e}_3$ coincides with the spin direction in the ground state). The left-hand sides of (4) and (5) are the coefficients of $s_{-}(\mathbf{x})$ and $s_{-}(\mathbf{x+a})\, s_{-}(\mathbf{x})$ in the Schrödinger equation, respectively. We note that $\mathbf{n}^2(\mathbf{x}) = 1$ and $\mathbf{e}^2_{+}(\mathbf{x}) = 0$.

The energy of the constructed state is

$$E = -\sum_{x,a} J_{\alpha\beta}\, n^{\alpha}(x+a)\, n^{\beta}(x). \tag{8}$$

Equation (4) is the condition for the extremum of the energy (8). It is easy to verify that in the case of a one-dimensional chain Eq. (4) is the consequence of (5). For the sake of simplicity, we consider the case when two eigenvalues of $J_{\alpha\beta}$ coincide, $J_1 = J_2 = J < J_3$ (uniaxial anisotropy). In this case all the vectors $\mathbf{n}(\mathbf{x})$ that satisfy Eq. (4) lie in one plane and are determined by a single angle θ_x between the "3" axis and the vector $\mathbf{n}(\mathbf{x})$. Equations (5) reduce to a single difference equation

$$\cos(\theta_{x+a} - \theta_x) + \gamma \sin\theta_{x+a} \sin\theta_x = 1, \tag{9}$$

where $\gamma = (J_3 - J)/J$. An investigation of Eq. (9) shows that in addition to the trivial solution $\theta_x = 0$ or π there exists a solution of the domain-type wall in which θ_x varies monotonically from 0 to π, when x runs through the values from $-\infty$ to $+\infty$. The solutions of (9) are labeled by two indices: discrete, x_0, which indicates the number of the site at which $|\theta_x - \pi/2|$ is minimal, and θ_{x_0} is continuous. The energy is independent of these parameters.

In the case of a space with a large number of dimensions, Eqs. (4) and (5) are incompatible. However, in the limiting case of a continuous isotropic medium, when the variation of $n(x)$ is over scales L larger than the lattice constant a, and $J_{\alpha\beta} = J\delta_{\alpha\beta}$, the Eqs. (5) are statisfied automatically, accurate to within a small quantity $\sim (a/L)^2$. With accuracy $\sim a/L$, the states constructed by us are stationary states of the Heisenberg Hamiltonian. Equation (4) reduces in this case to a differential equation. It can be obtained as the Euler-Lagrange equation for the energy functional

$$E = \frac{J}{2}\int d^2x \left(\frac{\partial n^{\alpha}}{\partial x_{\mu}}\right)^2; \qquad n^2(x) = 1. \tag{10}$$

The problem reduces to finding nontrivial extrema of the functional (9) of the classical field of directions $\mathbf{n}(\mathbf{x})$. Exact solutions of this problem were obtained in the two-dimensional case by Skyrme.[1]

We have thus constructed inhomogeneous quantum states of a Heisenberg Hamiltonian (solitons). The coordinate x_0 characterizing these states commutes with the Hamiltonian. It follows therefore that H does not depend on the momentum p conjugate to the coordinate x_0. The energy of a state with a definite momentum is likewise independent of the momentum. The soliton velocity is therefore equal to zero. The immobility of the soliton is caused by the conservation of two important characteristics of the system, namely the projection of the total spin and the topological characteristic of the lattice, the degree of mapping.[2]

In the two- and three-dimensional cases the degree of mapping has meaning only in the continuous limit. For a lattice system, such a quantity can be conserved only approximately. Therefore solitons of large dimensions will be long-lived on the lattice.

Although the solitons are separated by an energy gap from the ground state, they can play an essential role in thermodynamics and kinetics at low temperatures, owing to the large statistical weight.

[1] T. H. R. Skyrme, Proc. Roy. Soc. **A247**, 260 (1958); J. K. Perring and T. H. R. Skryme, Nucl. Phys. **31**, 550 (1962).

[2] A. M. Polyakov, Zh. Eksp. Teor. Fiz. **68**, 1975 (1975) [Sov. Phys.-JETP **41**, No. 6 (1975)].