Hyperspherical trees and 3nj coefficients

G. I. Kuznetsov and Ya. A. Smorodinskiĭ

I. V. Kurchatov Institute of Atomic Energy
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Simple rules are presented for the construction of basis hyperspheric wave functions for systems with many degrees of freedom. A connection is observed between the tree conversion coefficients and the Racah coefficients (and the $3nj$ symbols).

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The rules for writing down the solutions of the Laplace equations in multidimensional spheres (the tree method) and the rules for the graphical representation of the $3nj$ coefficients were formulated earlier in.[1,2] Each node is characterized by a variable-separation constant α_i and by a number of higher nodes; $S_{\alpha_i} = N_i - 2$, where N_i is the number of dimensions of the subspace, and $\alpha_i(\alpha_i + S_{\alpha_i})$ is the square of the hyper-angular momentum. The integration was carried out over an N_i-3-dimensional sphere. These rules can be simplified by describing the node not by two but by only one parameter and by integrating with a measure corresponding to a three-dimensional sphere 0(4)/0(3). The node is characterized by j_{α_i}, with $2j_{\alpha_i} + 1 = \alpha_i + S_{\alpha_i}/2$. We can then write for the multiplier (each node introduces its own multiplier into the solution) the following expression represented in terms of a Jacobi polynomial:

$$\bigvee_{j}^{j_1\ j_2} = N^{-1/2}(j_1 j_2 j)(1-x)^{\frac{2j_2+1}{2}}(1+x)^{\frac{2j_1+1}{2}} P_{j-j_1-j_2-1}^{2j_2+1,\,2j_1+1}(x) \quad (1)$$

where

$$N(j_1 j_2 j) = \frac{2^{2j_1+2j_2+3}\,\Gamma(j+j_1-j_2+1)\,\Gamma(j-j_1+j_2+1)}{(2j+1)\,\Gamma(j-j_1-j_2)\,\Gamma(j+j_1+j_2+2)}.$$

This solution, including the norm, is none other than the Wigner function[3] $d^j_{j_1+j_2+1,\,j_1-j_2}(x)$, normalized to unity on the 3-sphere; this function admits of an analytic continuation to quarter-integer (at odds S_{α_i}) values of j.

A transition from one cluster function to another corresponds to a "transplanting" of a branch of the type (see[4,5]):

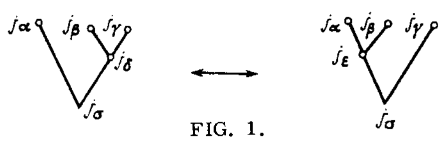

FIG. 1.

The matrix of this transition is the overlap integral between the functions corresponding to the left-hand and right-hand trees. It is determined by a hypergeometric function ${}_4F_3$ of unit argument.[4,5] Using the connection between the $6j$-symbols and ${}_3F_4(\ldots., \ldots 1)$, their symmetry with respect to the substitution $j \to -j - 1$, and the symmetry determined by formulas (22.1) and (22.22) of[6], we can represent the matrix of the transformation between the trees (Fig. 1) in the form

$$\left\| \begin{matrix} j_1 j_2 j_3 \\ j_{12} j\, j_{23} \end{matrix} \right\| = i\,(-)^{j_1+j_2+j_3}\sqrt{(2j_{12}+1)(2j_{23}+1)}$$

$$\times \begin{Bmatrix} j_1 j_2 j_{12} \\ j_3 j\, j_{23} \end{Bmatrix} \frac{1}{2}\left[\Phi^{j_{12} j_{23}}_{j_1 j_3 j} + \Phi^{j_{12} \rightleftarrows j_{23}}_{j_1 \rightleftarrows j_3 j}\right], \quad (2)$$

where

$$\Phi^{j_{12} j_{23}}_{j_1 j_3 j} = (-)^{j_{12}+j_3}\left\{\frac{\sin(j-j_1-j_{23})\pi\,(-)^{j-j_1-j_{23}}\sin 2\pi j}{\sin 2\pi j_{12}\,\sin 2\pi j_3}\right\},$$

$$\begin{Bmatrix} j_1 j_2 j_{12} \\ j_3 j\, j_{23} \end{Bmatrix} - 6j\text{-symbol}$$

Thus, the $6j$-symbol is connected with rotations of the 3-sphere. In the transplantations one encounters also schemes with one or two open forks:

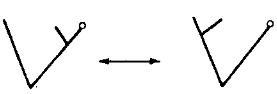

FIG. 2.

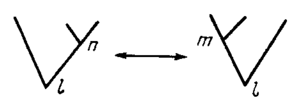

FIG. 3.

The matrices of these transitions are expressed in terms of ${}_3F_2(\ldots, \ldots 1)$ and ${}_2F_1$, i.e., they are either $3j$-symbols or d^l_{mn} functions of the angle $\pi/2$.[4,5] On the other hand, these matrices are particular cases of the matrix (2). It can be shown that the free ends can be regarded as nodes with α, $\beta = 0$ and S_α, $S_\beta = -1$ i.e., we can assume that j_α and j_β are equal to $-\frac{3}{4}$.

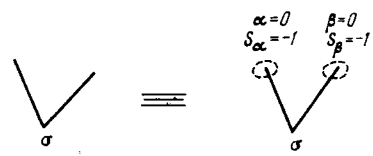

FIG. 4.

This is seen, for example, from the formula for the Chebyshev polynomial

$$T_n(\cos\phi) = \frac{2^{2n}(n!)^2}{(2n)!} P_n^{-1/2,\,-1/2}(\cos\phi) = \cos\phi\,.$$

Since the Racah and the Clebsch-Gordan coefficients are expressed respectively in terms of ${}_4F_3$ and ${}_3F_2$ of

..., it is of interest to show that the $9j$-symbol is determined by the function ${}_5F_4(\ldots 1)$. It is clear from the foregoing that the coefficient of transformation of the hyperspherical functions are the analogs of $3nj$-coefficients. For example, the problem of symmetrization of the wave function of three particles leads to $9j$-coefficients from which one makes up the matrix element of the permutation of two axes in 4-dimensional space. Thus, useful relations appear between the matrix elements of rotations through an angle π in multidimensional space and the algebra of $3nj$ coefficients. It is of interest to trace this connection for an arbitrary angle and, in particular, ascertain the connection between the generalized hyperspherical functions with the $3nj$ coefficients for $n > 2$ (the representations for the $3nj$ coefficients can be found, for example, in[6,7]).

A more detailed paper will be published later.

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