Dispersion of orbital waves in the A phase of superfluid He^3

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A linear dispersion law is derived for orbital waves in the A phase of He^3.

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Orbital waves are collective excitations connected with violation of invariance to rotations in the A phase of He^3. The dispersion of orbital waves, which was calculated[1] near T_c without allowance for dissipative processes, is of the form $\omega \sim \rho_s q^2/mL$, where **q** is the wave vector, ρ_s is the superfluid density, and L is the density of the spontaneous orbital momentum of the A phase. The value of L as determined by Cross[2] as well as by the present author[3] turned out to be of the order of $\rho_s(T_c/\epsilon_F)^2$. Allowance for the dissipative processes in the calculation of Cross and Anderson[4] has shown that orbital waves with quadratic dispersion are strongly damped because of the smallness of L in comparison with the coefficient μ of the viscous forces ($\mu = (\pi^2/128)(N_F\tau\Delta^2(T)/T_c$ near T_c, where τ is the relaxation time in a normal Fermi liquid and N_F is the density of states on the Fermi surface). We shall show *that in the A phase there is a temperature region near* T_c, where the quadratic dispersion of the orbital waves turns into linear at sufficiently high frequencies (but within the limit $\omega \ll 1/\tau$ of the validity of hydrodynamics), and the damping of these waves becomes weak.

The quasiequilibrium state of the A phase of He^3 is characterized by the local rotation of a triplet of vectors (Δ', $\vec{\Delta}''$, $\mathbf{l}$) which describes the order parameter of the Anderson-Morel axial state, relative to the equilibrium position, through an angle $\theta(\mathbf{r}, t)$. The equation of motion for θ follows from the equation for the change of the angular momentum $\delta\mathbf{L}$

$$\delta\dot{\mathbf{L}} = -\frac{\partial F}{\partial \vec{\theta}} - \mu\dot{\vec{\theta}}_\perp, \qquad (1)$$

where F is the free energy, and the second term in the right-hand side is the moment of the viscous forces, calculated in[4], and $\vec{\theta}_\perp$ is the projection of $\vec{\theta}$ on an axis perpendicular to $\mathbf{l}$. The change $\delta\mathbf{L}$ of the angular momentum is due both to rotation of the spontaneous angular momentum $\mathbf{L}$ through an angle $\vec{\theta}$, and to the appearance of an induced angular momentum proportional to the angular velocity of rotation $\dot{\vec{\theta}}$:

$$\delta\mathbf{L} = [\vec{\theta}\mathbf{L}] + \overleftrightarrow{\chi}\dot{\vec{\theta}}. \qquad (2)^1$$

Let us determine the tensor $\overleftrightarrow{\chi}$ in the weak-coupling approximation. We note to this end that the rotation of the vectors triplet (Δ', Δ'', $\mathbf{l}$) through an angle $\vec{\theta}$ leads to the following change in the phase of the order parameter:

$$\phi_\mathbf{k} = \operatorname{Im}\frac{\delta\Delta_\mathbf{k}}{\Delta_\mathbf{k}} = \operatorname{Im}\frac{(\mathbf{k},[\vec{\theta}\vec{\Delta}])}{(\mathbf{k}\vec{\Delta})} = \frac{[\vec{\theta}\mathbf{k}][\mathbf{k}\,\mathbf{l}]}{[\mathbf{k}\,\mathbf{l}]^2}. \qquad (3)$$

This change of phase is equivalent to adding to the Hamiltonian the term

$$F_\phi = \int d^3r \sum_\mathbf{k}\left[\frac{1}{2}\dot{\phi}_\mathbf{k}(\mathbf{r}, t) + \frac{1}{2}(\mathbf{k}\vec{\nabla})\phi_\mathbf{k}(\mathbf{r}, t)\right] n_\mathbf{k}(\mathbf{r}, t) \qquad (4)$$

just as in the ordinary superfluidity, except that ϕ depends on the direction of the momentum $\mathbf{k}$. The coefficient of $-\dot{\vec{\theta}}$ in the right-hand side of (4) has the meaning of the angular momentum, so that the variation of the density of the angular momentum is

$$\delta\mathbf{L}(\mathbf{r}, t) = \sum_\mathbf{k} \mathbf{l}_\mathbf{k}\delta n_\mathbf{k}(\mathbf{r}, t), \qquad \mathbf{l}_\mathbf{k} = \frac{[\mathbf{k}[\mathbf{l}\,\mathbf{k}]]}{2[\mathbf{k}\,\mathbf{l}]^2}. \qquad (5)$$

Under local equilibrium, the particle distribution function is $n_k = \frac{1}{2} - (\frac{1}{2})(\xi_k/E_k)\tanh(E_k/2T)$, where $E_k = (\xi_k^2 + |\Delta_k|^2)^{1/2}$ is the quasiparticle energy (no account is taken here of terms odd in $\mathbf{k}$, which do not contribute to $\delta\mathbf{L}$).

Substituting

$$\delta n_\mathbf{k} = \frac{\partial n_\mathbf{k}}{\partial \xi_\mathbf{k}}\frac{\dot{\phi}_\mathbf{k}}{2} + \frac{\partial n_\mathbf{k}}{\partial|\Delta_\mathbf{k}|^2}\delta|\Delta_\mathbf{k}|^2$$

in (5) and recognizing that variation of the modulus of the order parameter is $\delta|\Delta_k|^2 = 2\Delta^2(T)(\mathbf{k}\cdot\mathbf{l})(\vec{\theta}, [\mathbf{k}\times\mathbf{l}])$, we obtain for $\delta\mathbf{L}$ expression (2) with the following values of the tensor $\overleftrightarrow{\chi}$ and the vector $\mathbf{L}$:

$$\chi_{ij} = \frac{N_F}{4}\int\frac{d\Omega}{4\pi}\frac{[\mathbf{k}[\mathbf{k}\mathbf{l}]]_i[\mathbf{k}[\mathbf{k}\mathbf{l}]]_j}{[\mathbf{k}\mathbf{l}]^4} \qquad (6)$$

$$\mathbf{L} = -\mathbf{l}\frac{\Delta^2(T)}{4}\sum_\mathbf{k}(\hat{\mathbf{k}}\mathbf{l})^2\frac{\partial}{\partial\xi}\frac{\operatorname{th}\frac{E}{2T}}{E}, \qquad \hat{\mathbf{k}} = \frac{\mathbf{k}}{|\mathbf{k}|}. \qquad (7)$$

The last equation was obtained in[3] with the aid of a matrix kinetic equation.[5] Let us consider the components of the tensor $\overleftrightarrow{\chi}$. The longitudinal part of the tensor is $\chi_\parallel = N_F/4$. With the aid of this expression we can verify that the longitudinal component of Eq. (1) is a continuity equation. Indeed, the phase of the condensate is $\phi_0 = \theta_\parallel$, and therefore

$$-\frac{\partial F}{\partial\theta_\parallel} = \frac{\partial F}{\partial\phi_0} = -\frac{1}{2}\vec{\nabla}\mathbf{j}, \text{ a } \delta L_\parallel = -\frac{N_F}{4}\dot{\phi}_0 = \frac{1}{2}\delta\rho.$$

We note that such a dependence of $\delta L_\parallel$ on $\delta\rho$ could lead to the conclusion that the spontaneous angular momentum is $\mathbf{L} = \hbar\rho\mathbf{l}/2$, as proposed by Hall,[6] i.e., all the pairs have an orbital angular momentum $\mathbf{l}$ in the same direction. This conclusion does not contradict in principle the results of[3] and the present results, since the change of the angular momentum calculated in these studies is the result of a local deviation of the system from the equilibrium position. However, Hall's conclusion that it is precisely the angular momentum $\mathbf{L} = \hbar\rho\mathbf{l}/2$ which enters in expression (2) is incorrect, since Eqs. (1) and (2) with $T = 0$ can be obtained directly from the matrix kinetic equation with allowance for the self-consistent equation for the order parameter. In this case we obtain for $\overleftrightarrow{\chi}$ and $\mathbf{L}$ expressions (6) and (7).

We now consider the transverse component of the tensor

$$\chi_\perp = \frac{N_F}{8}\int\frac{d\Omega}{4\pi}\frac{1}{[\mathbf{k}\mathbf{l}]^2}.$$

The integral with respect to the angles diverges logarithmically. The divergence is caused by the fact that the expression for $\chi_\perp$ is no longer valid in the region of $\mathbf{k}$ where the gap $\Delta_k = \Delta(T)|[\mathbf{k}\times\mathbf{l}]|$ becomes comparable with the frequency ω. Therefore, restricting the integration to the angle region $|[\mathbf{k}\times\mathbf{l}]| > \omega/\Delta(T)$, we obtain

$$\chi_\perp \approx \frac{N_F}{8}\ln\frac{\Delta(T)}{\omega}. \qquad (8)$$

Calculation by the matrix kinetic equation method with $T = 0$ yields in place of $\ln(\Delta(0)/\omega)$ the complex quantitiy $\ln(\Delta(0)/\omega)+(\pi i/4)$. The imaginary part results from the possibility of excitation production at arbitrarily small ω, since the gap in the excitation spectrum vanishes at $\mathbf{k} \parallel \mathbf{l}$. If $\ln(\Delta(T)/\omega)\gg 1$, the contribution made by this imaginary increment to the damping of the orbital waves is weak. Let us determine the frequencies and temperatures at which we can neglect the viscous term $-i\omega\mu\vec{\theta}_\perp$ in Eq. (1). Comparing it with the dynamic term $\omega^2\chi_\perp\vec{\theta}$, we obtain the following bounds of the frequency

$$\frac{1}{\tau} >> \omega >> \frac{\mu}{\chi_\perp} \quad . \tag{9}$$

The left-hand inequality should ensure applicability of hydrodynamics. The inequalities (9) are compatible at $\Delta(T)\ll[(16/\pi^2)(T_c/\tau^2)\ln(\Delta(T)/\omega)]^{1/3}\sim 0.2T_c\ln^{1/3}(\Delta(T)/\omega)$, if τ is taken, just as in[4], from the data on spin diffusion ($\tau\sim 2\times10^{-13}/T^{\circ 2}$sec). Consequently weakly damped waves are possible at $T_c-T<10^{-2}T_c$. We write down the equations for the orbital waves in the fourth-sound regime ($\mathbf{v}_n=0$), using the expression for the free energy F from[1]

$$\omega^2\chi_\perp\vec{\theta}_\perp = \frac{\rho_{s\parallel}}{8m}\,[\,2q_\perp q_\parallel\phi_0 + (2q_\parallel^2 + q^2)\vec{\theta}_\perp]\,,$$
$$\omega^2\chi_\parallel\phi_0 = \frac{\rho_{s\parallel}}{4m}](q_\parallel^2+2q_\perp^2)\phi_0 + q_\parallel q_\perp\vec{\theta}_\perp]\,. \tag{10}$$

We see that the orbital waves are coupled with the density oscillations. They can be decoupled only if $\ln(\Delta(T)/\omega)\gg 1$, i.e., if $\chi_\perp\gg\chi_\parallel$. In this case the dispersion of the orbital waves is given by

$$\omega^2 = \frac{\rho_{s\parallel}}{8m\chi_\perp}\,q^2\,\frac{2q_\perp^2+3q_\parallel^2}{2q_\perp^2+q_\parallel^2} \quad . \tag{11}$$

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1) $[\vec{\theta}\mathbf{L}]\equiv\vec{\theta}\times\mathbf{L}$.

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