

Analog of Landau damping the problem of sound-wave propagation in a liquid with gas bubbles

D. D. Ryutov

Novosibirsk State University

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It is shown that a small initial perturbation in a liquid with gas bubbles attenuates to the end even in the absence of any dissipative processes, owing to the transition of the sound-wave energy into the energy of the natural oscillations of the bubbles. Other examples are presented of macroscopic systems in which this effect, which is analogous to the Landau damping in plasma physics, is manifest.

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We consider a liquid with gas bubbles that are randomly but on the average uniformly distributed in it. The characteristic bubble radius R is assumed small in comparison with the distance l between bubbles; the volume fraction a occupied by the bubbles is small, $a \sim (R/l)^3 \ll 1$. In the investigation of the propagation of sound waves in such a medium, with wavelength $\lambda \gg l$, the medium can be regarded as "continuous" and the macroscopic equations can be used for quantities averaged over a volume with dimensions small compared

with λ but large compared with l . With this approach, a plane sound wave of small amplitude is described by the equation (see, e.g., the review^[1]):

$$\partial^2 \delta \bar{p} / \partial t^2 = \partial^2 \delta \bar{p} / \partial x^2, \quad (1)$$

where $\delta_{\bar{p}}$ and $\delta_{\bar{\rho}}$ are the perturbations of the (average) density and pressure. The connection between $\delta_{\bar{p}}$ and $\delta_{\bar{\rho}}$ is given by

$$\delta \bar{p} = -\frac{1}{s_{11q}^2} - 4\pi\rho_{liq} \int_0^\infty \xi_r r^2 f(r) dr, \quad (2)$$

where s_{11q} is the speed of sound in the pure liquid, ρ_{11q} is the density of the pure liquid, $f(r)$ is the bubble-size distribution function ($f(r)dr$ is the number of bubbles having a radius in the interval between r and $r+dr$ per unit volume), and ξ_r is the change of the radius of a certain bubble under the influence of the variable pressure (since this change, at a given δp , depends on r , we label ξ with the subscript r). The time variation of the bubble radius is described by the equation

$$\frac{\partial^2 \xi_r}{\partial t^2} + \omega_0^2(r) \xi_r = -\frac{\partial \bar{p}}{\rho_{liq} r}, \quad (3)$$

where $\omega_0(r)$ is the natural frequency of the radial oscillations of the bubble. In the widespread case of isothermal variation of the bubble volume (see^[1]) we have $\omega_0^2 = 3p/\rho_{11q}r^2$, where p is the unperturbed pressure in the medium. We note the relation $s_g^2 \sim p/\rho_g$, where s_g is the speed of sound in the gas filling the bubbles and ρ_g is the density of this gas. It follows from this relation that the parameter $\epsilon = p/\rho_{11q}s_{11q}^2$ is of the order of $(\rho_g/\rho_{11q})(s_g/s_{11q})^2$, and is consequently small in comparison with unity.

No dissipative processes are taken into account in Eqs. (1)–(3). Nonetheless, as will be shown below, the sound wave attenuates to the end in this case, owing to a process analogous to Landau damping of electrostatic plasma oscillations.^[2] The damping decrement is easiest to determine in the case when the fraction a of the volume occupied by the bubbles is small in comparison with ϵ . Then the second term in the right-hand side of (2) is small in comparison with the first and can be regarded as a perturbation. In the zeroth approximation in a we obtain from (1) and (2), for perturbations in the form $\exp[-i\omega t + ikx]$, the usual dispersion equation $\omega = ks_{11q}$. In the next order of approximation, we can easily obtain the correction to the frequency

$$\omega = ks_{11q} \left[1 + 2\pi s_{11q}^2 \int_0^\infty \frac{r f(r) dr}{(ks_{11q} + iO)^2 - \omega_0^2(r)} \right]$$

In the last term we write $ks_{11q} + iO$ in place of ks_{11q} , recognizing that the perturbation must vanish as $t \rightarrow \infty$. The small contribution of the bubbles to the real part of the frequency is of no interest; the decrement can be obtained with the aid of the known formula $\text{Im}(x + iO)^{-1} = -\pi\delta(x)$. The result takes the form

$$\gamma = \pi^2 s_{11q}^2 \int_0^\infty r f(r) \delta[ks_{11q} - \omega_0(r)] dr. \quad (4)$$

For waves with frequencies in the interval of the resonant frequencies of the bubbles (i.e., with $k \sim \sqrt{\epsilon}/R$), the estimate $\gamma/ks_{11q} \sim \alpha/\epsilon$ is valid (we took account of the fact that $f \sim \alpha/R^4$), if the scatter ΔR of the bubble radii relative to the mean value is small, $\Delta R \ll R$, then the damping decrement differs from zero only in a narrow interval of the values of the wave vector around $k = \omega_0(R)/s_{11q}$, but on the other hand its value increases:

last estimate of the decrement is valid for not too "narrow" distributions, such that the resonant-frequency interval $\Delta\omega \sim \omega_0(R)\Delta R/R$ is still large in comparison with γ . The result is the following limitation on $\Delta R/R$: $\Delta R/R \gg \sqrt{\alpha/\epsilon}$. The opposite limiting case calls for a separate analysis.

We have neglected above (particularly in Eq. (3)) the damping of the bubble oscillations due to emission of spherical sound waves by the latter. The radiative damping decrement γ_{rad} is of the order of $\omega_0\sqrt{\epsilon}$. Our approximation is valid if $\gamma \gg \gamma_{rad}$, i.e., $\alpha \gg \epsilon^{3/2}$. Under this condition, the wave energy is transferred within a time $\sim \gamma^{-1}$ to the natural oscillations of the bubbles and these oscillations attenuate, after a much longer time $\sim \gamma_{rad}^{-1}$, and generate secondary scattered waves.

If the bubble concentration α exceeds ϵ , then the fluence of the bubbles on the dispersion becomes decisive. To investigate the temporal evolution of the spatially-harmonic ($\propto \exp[ikx]$) initial perturbation in this case it is necessary to use a Laplace transform with respect to time (cf. ^[2]). Calculations perfectly similar to those in^[2] show that as $t \rightarrow \infty$ the dependence of the perturbation on the time are determined by the factor $\exp(qt)$, where q is the root, closest to the imaginary axis, of the equation^[1] $k^2 s_{11q}^2 + q^2 [1 + F(q)] = 0$. The function $F(q)$ entering in this equation is defined as follows: in the right-hand half-plane of the complex variable q it is equal to

$$4\pi s_{11q}^2 \int_0^\infty \frac{r f(r) dr}{q^2 + \omega_0^2(r)},$$

and in the left half-plane it is defined as the analytic continuation of this integral. The characteristic phase velocity of the acoustic perturbations at $\alpha \gg \epsilon$ is estimated at $s_{11q}\sqrt{\epsilon/\alpha}$. Accordingly perturbations with $ks_{11q} \sim \omega_0(R)\sqrt{\alpha/\epsilon}$ "resonate" with the bubbles and are strongly attenuated. For such perturbations we have $\text{Re}q \sim \text{Im}q \sim \omega_0(R)$. We note that, in analogy with the case of Langmuir oscillations, the exponential law for the decrease of the perturbations holds only asymptotically, as $t \rightarrow \infty$.

Landau damping manifests itself in the boundary-value problem of spatial absorption of a wave of the time $\exp(-i\omega t)$. The spatial absorption was related in^[3] to radiative damping of the bubble oscillation, but this is not compatible with the physical meaning of the problem, since the time necessary to establish a stationary amplitude distribution is of the order of $(\text{Re}q)^{-1}$ and is small in comparison with the reciprocal radiative-damping decrement (which can be estimated at $\alpha \gg \epsilon$ and $\Delta R \sim R$ from the formula $\gamma_{rad} \sim \omega_0\sqrt{\alpha}$). The oscillation amplitude decreases towards the interior of the medium like $\exp(-\kappa x)$, where κ is estimated for oscillations with $\omega \sim \omega_0(R)$ from the formula $\kappa \sim (\alpha/\epsilon)^{1/2} \times \omega_0(R)/s_{11q}$ (at $\Delta R \sim R$).

The Landau damping can manifest itself also in other macroscopic systems with random inhomogeneities. We present two examples.

1) Plasma with "magnetic filaments," i.e., a plasma where the magnetic field is concentrated inside narrow tubes that are far from one another (this situation seems to be realized in the solar chromosphere, see^[4]). The long sound waves are absorbed in this system because of resonant excitation of flexural oscillations of the filaments (an analogous property would be possessed by a medium consisting of a liquid in which ordinary elastic filaments are stretched).

2) An isotropic elastic body in which the speed of the longitudinal sound appreciably exceeds that of the transverse sound, and where there are randomly arranged small spherical cavities. The long-wave longitudinal sound attenuates in such a system because of excitation of weakly-damped (see^[5]) oscillations of the cavities.

In addition to direct applications, the results can be of interest from the point of view of the use of relatively simple macroscopic systems for the simulation of plasma phenomena.

¹⁾Expression (4) is, of course, a particular case of this result.

¹G. G. K. Batchelor, in: *Mekhanika (Mechanics)*, No. 3, p. 65, 1968.

²L. D. Landau, *Zh. Eksp. Teor. Fiz.* **16**, 574 (1946):

³E. Meyer and E. Skudrzyk, *Akustische Beihefte* **3**, 434 (1953).

⁴G. A. Chapman, *Astrophys. J.* **191**, 255 (1974).

⁵L. D. Landau and E. M. Lifshitz, *Teoriya uprugosti (Theory of Elasticity)*, Nauka, 1965, p. 136 [Addison-Wesley].