Concerning the effect of the decay phonon spectrum on the viscosity of helium II

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We calculate the viscosity coefficient of a phonon gas having a decay spectrum.

The viscosity (as well as other kinetic phenomena) in superfluid helium II at low temperatures (below 0.5 °K) is determined by the phonon interaction, i.e., only by the interaction of only those elementary excitations that are present in that case. If the phonons spectrum is stable, phonon decay is impossible and the most probable is the scattering of phonons by phonons (the four-phonon process). Then, if we disregard the dispersion of the phonon spectrum (i.e., the deviation from linearity), then the phonon-phonon scattering probability, in second order of perturbation theory in the cubic anharmonic terms, diverges at small scattering angles. A detailed analysis of this process shows that it leads to a rapid establishment of the equilibrium of the phonons moving in a given direction. Such collinear processes, however, are of no importance for phenomena such as viscosity and thermal conductivity. On the other hand, an exact solution of the kinetic equation for the viscosity problem shows that four-phonon processes (even if dispersion is not taken into account) lead to a finite result in the collision integral. The viscosity of helium II is determined precisely by the corresponding characteristic time $1/\tau_4$, which characterizes the angular diffusion of the nonequilibrium phonons. According to[1], this time is given by

$$\frac{1}{\tau_4} = \frac{9 \cdot 13!\,(u+1)^4}{2^{13}(2\pi)^7 \hbar^7 \rho^2 c}\left(\frac{kT}{c}\right)^9, \quad u = \frac{\rho}{c}\frac{\partial c}{\partial \rho}. \quad (1)$$

The viscosity coefficient is then

$$\eta_{ph} \approx \frac{1}{5}\,\rho_{nph}\,c^2\tau_4 \sim T^{-5}. \quad (2)$$

At the present time it can apparently be regarded as established that at low pressures (below 15 atm) the phonon part of the energy spectrum of superfluid helium is of the decay type,[2] i.e.,

$$\epsilon = cp\,(1 + \gamma p^2), \text{ where } \gamma > 0. \quad (3)$$

Phonon decay (the three-phonon process) therefore becomes essential for the kinetic phenomena.

The theory developed in[1,3] makes it possible, as we shall presently show, to calculate readily the characteristic time of phonon angular diffusion as a result of the three-phonon process. It turns out here that the characteristic time $1/\tau_3$ has the same temperature and the same order of magnitude as $1/\tau_4$. Naturally, the viscosity coefficient η_{ph} will also have the same temperature dependence as previously obtained in[1].

Thus, let us calculate the viscosity of a phonon gas that has the decay spectrum (3). Choosing a coordinate system with polar axis directed along the vector of the macroscopic velocity **u**, we write down the kinetic equation[1]

$$n_0(n_0+1)\frac{cp}{kT}\frac{\partial u}{\partial x}\cos\theta\sin\theta\cos\phi = I_3. \qquad (4)$$

The collision integral I_3 is a sum of two integrals describing respectively the number of phonons lost through coalescence with others and through decay:

$$I_+ = -\frac{c}{(2\pi\hbar)^3}\int do_+[n_1 n_2(n_3+1)-n_3(n_1+1)(n_2+1)]d\mathbf{p}_2.$$
$$\qquad (5)$$
$$I_- = -\frac{c}{(2\pi\hbar)^3}\int do_-'\,\{n_1(n_2+1)(n_3+1)-n_3 n_2(n_1+1)\}d\mathbf{p}_2.$$

All the notation here and below is standard. Each of the integrals I_+ and I_-, when expanded in terms of the small quantity γp^2 (p is a certain momentum), or equivalently in terms of the small angles between the phonons, is proportional to the quantity γp^2. It turns out, however, that $I_+ = I_-$ accurate to γp^2. Therefore the expansion of the integral I_3 begin with a term of order $(\gamma p^2)^2$.

Equilibrium with respect to the number of phonons moving in a given direction establishes a parallel three-phonon process with a rate

$$\frac{1}{r_{11}} \approx \frac{(u+1)^2\zeta(2)}{2\pi\hbar^4\rho}x^2\left(\frac{kT}{c}\right)^5 \qquad \begin{array}{l} x = 7-8 \\[4pt] \zeta(2)=1.3. \end{array} \qquad (6)$$

Taking the symmetry of (1) into account, the deviation of the distribution function from equilibrium will be sought in the form

$$\delta n = -n_0(n_0+1)(\epsilon\nu P_2(\cos\theta)-p\beta P_2(\cos\theta)), \qquad (7)$$

where $P_2(x)$ is a Legendre polynomial.

We note that since ϵ depends nonlinearly on the momentum, it follows that the coefficient β cannot be included in the coefficient ν, as could be done in the case of linear dispersion. All the three-dimensional processes occur in a narrow cone of directions around a phonon with a specified momentum $\mathbf{p}_1$. Therefore, at our accuracy, we can assume that the azimuthal angle of the phonon $\mathbf{p}_2$ coincides with the azimuthal angle of the phonon $\mathbf{p}_1$, i.e., the main contribution to the integral is made by phonons in the momentum plane in which the vector $\mathbf{u}$ lies. This differs from the case of

four-phonon processes, where the azimuth ϕ_2 of the phonon $\mathbf{p}_2$ can be arbitrary.

Taking all the foregoing into account, expanding I_3 up to $(\gamma p^2)^2$, and then integrating (1) with respect to $\epsilon p^2 dp$ and $p^3 dp$, we obtain the connection between ν and β:

$$\beta = \frac{3}{2}\nu. \qquad (8)$$

Then (4) is rewritten in the form

$$\frac{4\pi^4}{15}\frac{\partial u}{\partial x} = -\frac{\nu}{r_3}; \qquad \frac{1}{r_3} = b\frac{(u+1)^2\gamma^2 T^9}{\hbar^4\rho c^9} \qquad b = 0.53\cdot10^7 \qquad (9)$$

The viscosity is therefore

$$\eta = \frac{2\pi^5}{(15)^3}c\left(\frac{kT}{c}\right)^4\frac{r_3}{\hbar^3}. \qquad (10)$$

Thus, the temperature dependence of the viscosity, with the employed dispersion law, coincides with the temperature dependence of the four-phonon viscosity, but the numerical coefficient depends on the value of γ.[1)]

The ratio τ_4/τ_3 contains the parameter $\hbar^3\rho\gamma^2 c\ll1$, and only the numerical factors make the times τ_4 and τ_3 of the same order of magnitude. The fact that the relaxation times for the three-phonon and four-phonon processes are of the same order of magnitude and have the same temperature dependence indicates that the role of the multiphonon processes deserves a special detailed study.

[1)]Analogous results were recently obtained in[4] by approximate methods, and in[5], where a very cumbersome method, using the transverse-relaxation operator, was employed.

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