Example of nontrivial interaction of solitons in two-dimensional classical field theory

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It is shown that in the model of the principal chiral field on the *SU(N)* group with $N \geq 3$ there are soliton solutions with nontrivial interaction (mutual transformation) of solitons.

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The first example of a fully integrable nonlinear field theory in two-dimensional space-time was the sine-Gordon model.[1] Subsequently, the method of the reciprocal scattering problem has made it possible to integrate the massive Thirring model[2] and the complex generalization of the sine-Gordon equation[3] introduced by Pohlmeyer.[4] Qualitatively, the solutions of these problems have much in common. Namely, a natural subdivision of the initial condition, into a soliton part and a nonsoliton part, takes place in the course of the evolution. The interaction between the solitons is trivial, i.e., it leads only to a shift of the gravity centers of their phases, while multiple production processes are forbidden by the conservation laws. A nontrivial interaction of solitons (decay of the soliton of one wave into two solitons of other waves) was observed for the first time in the three-wave problem.[5] The question of the existence of a nontrivial interaction in relativistically invariant models remains open to this day. In this article we show that in the model of the principal chiral field[1)] on the group SU(N) with $N \geqslant 3$, there are soliton solutions with nontrivial soliton interaction.

Let an element of the group $g(x,t) \in \mathrm{SU}(N)$ be specified at each point of space, meaning a unitary unimodular matrix in N-dimensional "isotopic" space. The Lagrangian

$$L = \frac{1}{2} \int \mathrm{Sp}\, (\partial_\mu g \partial^\mu g^+) dx \tag{1}$$

and the constraint $gg^+ = I$ correspond to an equation of motion

$$\partial_\mu \partial^\mu g - \partial_\mu g g^+ \partial^\mu g = 0 . \tag{2}$$

In the case $g \in \mathrm{SU}(r)$, Eq. (2) coincides with the equation for the **n** field on a three-dimensional sphere, and consequently reduces to the complex sine-Gordon equation.[4] We introduce the Hermitian fields A_μ belonging to the Lie algebra

$$A_\mu = i\partial_\mu g g^+. \tag{3}$$

From (2) and from the constraint it follows that A_μ satisfies the system

$$\begin{aligned} &\partial_\mu A^\mu = 0 \\ &i\partial_\mu A_\nu - i\partial_\nu A_\mu + [A_\mu A_\nu] = 0. \end{aligned} \tag{4}$$

In the conical variables $\eta=(t+x)/2, \xi=(t-x)/2$ and $A=A_0-A_1$, $B=A_0+A_1$ the system (4) takes the form

$$\begin{aligned} \partial_\eta A &= \frac{i}{2}[A, B], \\ \partial_\xi B &= \frac{i}{2}[B, A]. \end{aligned} \tag{5}$$

We note that the fields A_μ can be interpreted as currents generated by the symmetry of the Lagrangian (1) relative to the group shifts $g \to hg$, where $h \in SU(N)$ is any constant matrix.

The application of the inverse-problem method to the system (3) is based on the fact that Eqs. (5) can be represented as the condition for the existence of a simultaneous solution of the system

$$i\partial_\xi \Psi - \frac{A}{\lambda+1}\Psi = 0, \tag{6}$$

$$i\partial_\eta \Psi + \frac{B}{\lambda-1}\Psi = 0 \tag{7}$$

at all values of λ. This can easily be verified by commuting the operators that act on Ψ in Eqs. (6) and (7). We note that any solution of the problem (6) and (7) is a solution of the system (5), and in addition, assuming

$$g = \Psi(\lambda, \eta, \xi)|_{\lambda=0}. \tag{8}$$

we obtain an exact solution of the problem. Let $A^0(\xi)$ and $B^0(\eta)$ be diagonal matrices. Then the system (6) and (7) is automatically compatible and has a general solution

$$\Psi_0(\lambda, \eta, \xi) = \exp\left[-i\int^{\xi}\frac{A^0(\xi')d\xi'}{\lambda+1} + i\int^{\eta}\frac{B^0(\eta')d\eta'}{\lambda-1}\right]. \tag{9}$$

We shall call the fields $A^0, B^0, g^0 = \Psi_0|_{\lambda=0}$ the vacuum solutions of Eqs. (5) and (2). To obtain a soliton solution, following[6], we represent Ψ in the form

$$\Psi = \Psi_1(\lambda, \eta, \xi)\Psi_0(\lambda, \eta, \xi), \tag{10}$$

where

$$\Psi_1(\lambda, \eta, \xi) = I - \frac{\lambda_0 - \bar{\lambda}_0}{\lambda - \bar{\lambda}_0} P(\eta, \xi) \tag{11}$$

and the P is a Hermitian projection opeator in isotropic space, while $\lambda_0{}^* = \lambda'_0 + i\lambda''_0$ is a complex number. It follows from (6) and (7) that the functions

$$\begin{aligned} \frac{A(\eta, \xi)}{\lambda+1} &= i\partial_\xi \Psi_1 \Psi_1^{-1} + \Psi_1 \frac{A^0(\xi)}{\lambda+1}\Psi_1^{-1} \\ \frac{B(\eta, \xi)}{\lambda-1} &= -i\partial_\eta \Psi_1 \Psi_1^{-1} - \Psi_1 \frac{B^0(\eta)}{\lambda-1}\Psi_1^{-1} \end{aligned} \tag{12}$$

have no singularities other than poles at the points $\lambda = -1$ and $\lambda = 1$, respectively. Substituting the representation (11) in expresion (12) and equating to zero the residues at the points $\lambda = \lambda_0$ and $\lambda = \bar{\lambda}_0$, we obtain the equations for the projector

$$\begin{aligned} (I - P)\left(i\partial_\xi - \frac{A^0(\xi)}{\lambda+1}\right)P &= 0 \\ (I - P)\left(i\partial_\eta + \frac{B^0(\eta)}{\lambda-1}\right)P &= 0. \end{aligned} \tag{13}$$

We confine ourselves here to the simplest situation, when there is only nonzero vector $\langle C^0| = \langle C^0{}_1 \ldots C^0{}_N|$ satisfying the condition

$$P(0, 0)|C^0\rangle = |C^0\rangle. \tag{14}$$

In this case

$$P(\eta, \xi) = \frac{\Psi_0(\lambda_0, \eta, \xi)|C^0\rangle\langle C^0|\Psi_0^+(\lambda_0, \eta, \xi)}{\langle C^0|\Psi_0^+(\lambda_0, \eta, \xi)\Psi_0(\lambda_0, \eta, \xi)|C^0\rangle} \tag{15}$$

is a solution of the system (13) with initial condition (14). Using the calculated projector (15), we get from (8), (10), (11), and (12) the soliton solution

$$g=\left(I+\frac{\lambda_0-\bar\lambda_0}{\bar\lambda_0}P(\eta,\xi)\right)g^{\circ}(\eta,\xi),$$

$$A_{pq}=a_p\delta_{pq}+4\frac{\lambda_0''}{|1+\lambda_0|^2}\left[i\frac{\lambda_0'+1}{2\lambda_0''}(a_p-a_q)-\frac{a_p+a_q}{2}+\sum_{s=1}^{N}a_sP_{ss}\right]P_{pq},$$

$$B_{pq}=b_p\delta_{pq}+4\frac{\lambda_0''}{|1-\lambda_0|^2}\left[i\frac{\lambda_0'-1}{2\lambda_0''}(b_p-b_q)-\frac{b_p+b_q}{2}+\sum_{s=1}^{N}b_sP_{ss}\right]P_{pq},$$

where

$$A^{\circ}_{pq}=a_p\delta_{pq},\quad B^{\circ}_{pq}=b_p\delta_{pq}.$$

Let us analyze this solution for a field with values in SU(3), assuming the matrices A^0 and B^0 to be constant. We shall show that it describes the decay of a soliton of one component of the field into two solitons of other components. The evolution of the solution is determined by the components $|P_{qq}|$ (this can be easily verified by calculating $|A_{pq}|$, $p\neq q$ or $|g_{pq}|$). We represent $|P_{pq}|$ in the form

$$|P_{pq}|=\frac{1}{2\operatorname{ch}(a_{pq}(x-v_{pq}t-x^{\circ}_{pq}))+\exp(\Gamma_{pq}t-\kappa_{pq})}\quad\text{at } p\neq q$$

$$|P_{ss}|=\frac{1}{1+2\operatorname{ch}(a_{pq}(x-v_{pq}t-x^{\circ}_{pq}))\exp(-\Gamma_{pq}t+\kappa_{pq})},\quad\text{at } s\neq p,q$$

where

$$a_{pq}=a_p-a_q,\ \beta_{pq}=\beta_p-\beta_q,\ v_{pq}=\beta_{pq}/a_{pq},\ x^{\circ}_{pq}=-\ln(C_p/C_q)/a_{pq},$$

$$\Gamma_{pq}=6\frac{a_p\beta_q-a_q\beta_p}{a_{pq}},\quad \kappa_{pq}=3(a_p+a_q)(x-v_{pq}t-x^{\circ}_{pq})-T_{pq},$$

$$T_{pq}=a_{sp}(x^{\circ}_{pq}+x^{\circ}_{ps})+a_{sq}(x^{\circ}_{pq}+x^{\circ}_{sq}),\quad a_p=\lambda_0''(b_p/|\lambda_0-1|^2+a_p/|\lambda_0+1|^2),$$

$$\beta_p=\lambda_0''(-b_p/|\lambda_0-1|^2+a_p/|\lambda_0+1|^2).$$

If the matrices A^0 and B^0 are linearly dependent (i.e., $a_p b_q - a_q b_p = 0$), then all $\Gamma_{pq}=0$ and the solution is a soliton on the decay boundary (with zero mass defect). In the nondegenerate case all the Γ_{pq} cannot be of the same sign. Let, for example, $\Gamma_{12} > 0 > \Gamma_{13}, \Gamma_{23}$, and then, as $t \to = -\infty$, there exists one soliton moving with velocity v_{12}:

$$|P_{12}| \to \frac{1}{2\operatorname{ch}(a_{12}(x - v_{12}t - x_{12}^{\circ}))}; \qquad P_{13}, P_{23} \to 0,$$

at $t \sim -T_{pq}/\Gamma_{pq}$ this soliton decays and as $t \to +\infty$ two solitons are produced, moving with velocities v_{13} and v_{23}, respectively:

$$P_{12} \to 0; \ |P_{13}| \to \frac{1}{2\operatorname{ch}(a_{13}(x - v_{13}t - x_{13}^{\circ}))}; \ P_{23} \to \frac{1}{2\operatorname{ch}(a_{23}(x - v_{23}t - x_{23}^{\circ}))}$$

The decay time is determined by the increment Γ_{pq}. If $\Gamma_{13}, \Gamma_{23} > 0 > \Gamma_{12}$ (this can be attained by another choice of the parameter λ_0), then the solution describes the coalescence of two solitons.

[1)]Following the terminology introduced by L.D. Faddeev, we define the principal chiral field as fields with values in the Lie group.

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