DECAY OF INITIAL DISCONTINUITY IN THE KORTEWEG-DE VRIES EQUATION

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As is well known, processes in nondissipative media with small nonlinearity and dispersion are described by the Korteweg-deVries equation

$$\frac{\partial \eta}{\partial t} + \eta \frac{\partial \eta}{\partial x} + \frac{\partial^3 \eta}{\partial x^3} = 0, \tag{1}$$

So far, however, (1) was used to investigate mainly the evolution of a perturbation that is concentrated at the initial instant in a finite region of space [1 - 3]. The purpose of the present paper is to solve a problem in which η experiences at the point $x = 0$ a finite jump, so that $\eta = \eta_0$ at $x < 0$ and $\eta = 0$ at $x > 0$. In the course of time this discontinuity changes into a broadening region occupied by the oscillations. At $\eta_0^{3/2} t \gg 1$, the dimension of this region is much larger than the oscillation wavelength, so that Witham's quasiclassical method can be used [4]. Equation (1) has a periodic solution

$$\begin{aligned} \eta(x,t) &= \frac{2a}{s^2}\, dn^2\left[\left(\frac{a}{6s^2}\right)^{1/2}(x - Vt),\, s\right] + \gamma \\ V &= \frac{2a}{3s^2}(2 - s^2) + \gamma, \end{aligned} \tag{2}$$

where $dn(u, s)$ is a Jacobi elliptic function with modulus s, $0 \le s \le 1$. The value $\bar{\eta}$ averaged over the period and the wave vector k are given by

$$\bar{\eta} = \gamma + \frac{2a\,E(s)}{s^2 K(s)}, \qquad k = \frac{\pi}{K(s)}\left(\frac{a}{6s^2}\right)^{1/2} \tag{3}$$

where K and E are complete elliptic integrals of the first and second kind, respectively.

We seek a solution in the form (2), assuming a, s, and γ to be slowly-varying functions of x and t. It is convenient to write the approximate equations for these functions, according to [4], by introducing three new quantities $r_3 > r_2 > r_1$:

$$r_2 - r_1 = 2a, \quad \frac{r_2 - r_1}{r_3 - r_1} = s^2, \quad r_1 + r_2 - r_3 = 2\gamma.$$

The equations for r_α are

$$\frac{\partial r_\alpha}{\partial t} + v_\alpha \frac{\partial r_\alpha}{\partial x} = 0, \qquad \alpha = 1, 2, 3; \tag{4}$$

where v_α are definite functions of r_α; we need only an expression for v_2:

$$v_2 = \frac{1}{6}(r_1 + r_2 + r_3) - \frac{2}{3} a \frac{(1 - s^2) K(s)}{E(s) - (1 - s^2) K(s)} .$$

Equations (4) do not contain a parameter with dimension of length, so that their solution, for our initial condition, should depend only on the ratio $r = x/t$. Then (4) reduces to

$$(v_a - r) \frac{dr_a}{dr} = 0.$$

We obtain the required solution by putting

$$r_1 = \text{const}, \quad r_3 = \text{const}, \quad v_2 = r. \tag{5}$$

The oscillations occupy a finite region in space, and on the leading front of this region, at $\tau = \tau_+$, we should have $r_2 = r_3$ and $s = 1$, i.e., $k = 0$. In other words, near the leading front the oscillations break up into a set of solitons. By virtue of the continuity at $\tau = \tau_+$, we should have $\bar{\eta} = 0$, from which we find with the aid of (3) that $r_1 = 0$. On the trailing edge, at $\tau = \tau_-$, the oscillation amplitude vanishes, i.e., $r_2 = r_1$ and $s = 0$. At this point $\bar{\eta} = \eta_0$, whence $r_3 = 2\eta_0$ and $a = \eta_0 s^2$. This means, in particular, that the amplitude of the leading soliton is equal to $2\eta_0$. Taking the foregoing into account, we have ultimately

$$\eta(x, t) = 2\eta_0 \, dn^2\left[\left(\frac{\eta_0}{6}\right)^{1/2}\left(x - \frac{1 + s^2}{3}\eta_0 t\right), s\right] - \eta_0 (1 - s^2), \tag{6}$$

with $s(x/t)$ determined by the equation

$$\frac{1 + s^2}{3} - \frac{2}{3} \frac{s^2 (1 - s^2) K(s)}{E(s) - (1 - s^2) K(s)} = \frac{x}{\eta_0 t} = \frac{r}{\eta_0} . \tag{7}$$

Formulas (6) and (7) in parametric form (the parameter is s) solve our problem, i.e., they determine $\eta(x, t)$ at $\eta_0^{3/2} \gg 1$. It follows from (7) that $\tau_- = -\eta_0$ and $\tau_+ = 2\eta_0/3$. As $\tau < \tau_-$ we have $\bar{\eta} = \eta_0$ and at $\tau > \tau_+$ we have $\eta = 0$. At $\tau \to \tau_+$ the mean value $\bar{\eta}$ behaves like $\ln^{-1}[1/(\tau_+ - \tau)]$, so that $\bar{\eta}$ tends to zero with infinite derivative on the leading front. The dependences of $\bar{\eta}$, k, and s^2 on τ for a unit discontinuity ($\eta_0 = 1$) is shown in Fig. 1 and the dependence of η on x at t = 50 and 100 is shown in Fig. 2. We see that the broadening of the oscillation region with

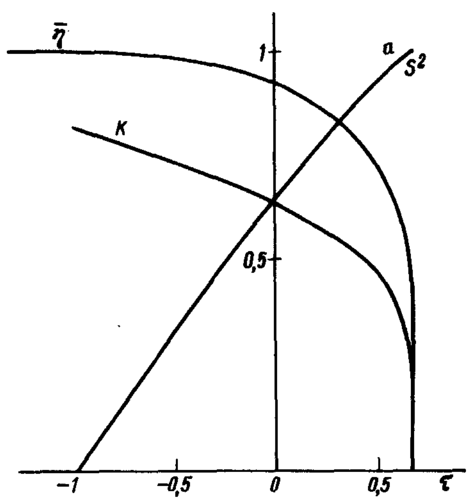

Fig. 1

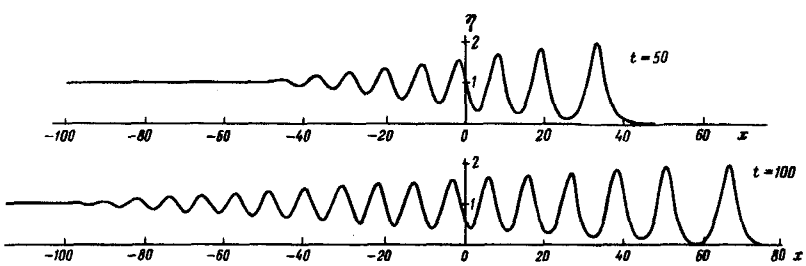

Fig. 2

increasing t is due to the increase in the number of oscillations, and the wavelength changes insignificantly.

The evolution of the discontinuity on the basis of the linearized equation (1) was considered in [5]. The formulas obtained there describe the initial stage of the process at $\eta_0^{3/2}t \ll 1$. The process should subsequently reach the asymptotic solution obtained above, in which the dispersion and nonlinearity effects are of the same order of magnitude. The decay of a small discontinuity in a plasma was experimentally investigated in [6]. The observed picture was qualitatively similar to that in Fig. 2.

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INFLUENCE OF SATURATION EFFECTS ON STIMULATING SCATTERING IN LASER HEATING OF A PLASMA

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We consider the reflection of powerful laser radiation from a plasma produced when laser pulses are focused on a solid target. We show that to calculate the reflection coefficient correctly it is necessary to take the nonlinear effects into account. In particular, saturation effects limit the reflection as a result of stimulated Thomson scattering and stimulating scattering by ions at the $2 \times 10^{11} - 10^{13}$ W/cm^2 level.

Recent papers [1, 2] discuss various stimulated-processes in a dense plasma produced by focusing powerful laser radiation on a solid target. Interest in these processes is connected with the problem of initiating thermonuclear reactions by heating a plasma with a laser [3]. It is feared in [1, 2] that at large fluxes the processes of stimulated scattering in peripheral plasma layers, with electron density $N_e \ll N_{cr} \approx 10^{21}$ cm^{-3}, where absorption at electron temperatures $kT_e \gtrsim 1$ keV is very small, can lead to a conversion of an appreciable fraction of the incident radiation into scattered waves that leave the plasma, i.e., to reflection of the optical energy.

We wish to call attention in the present communication to a number of important aspects not accounted for in [1, 2]. The most important of them are saturation processes that limit the exponential growth of the scattered waves. The fact that these processes must be taken into account can be seen from the following simple considerations. Let us assume that a plasma of density N is grouped in a lattice $N(x) = N[1 + \alpha\cos(4\pi x/\lambda)]$, where $\lambda = 2\pi c/\omega_L$ is the wavelength and ω_L the frequency of the laser radiation, and α is the depth of modulation. The dielectric constant of the plasma at the frequency ω_L is