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CRITICAL BEHAVIOR OF NONPLANAR ISING MODEL IN STRONG ANTIFERROMAGNETIC INTERACTION ALONG THE DIAGONALS

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It is shown that at small $\lambda = J_1/|J_2|$, where J_1 and J_2 are the constants of the interactions along the sides and along the diagonals, the shift of the transition temperature and the change of the critical exponents are proportional to λ^2. The exponents satisfy in this case the scaling hypothesis.

Only a few models in which a phase transition takes place have been solved exactly to date. These are, first, different variants of the planar Ising model with nearest-neighbor interaction [1], the Slater model and its modification (the so-called six-vertex or F model) [2], and finally the eight-vertex model without an external field or the Baxter model [3], of which the first two models are particular cases. The next in complexity is the nonplanar Ising model, i.e., the Ising model with intersection of the interactions. It is equivalent to the eight-vertex model in an external field; its solution has not been obtained so far, but it is possible to determine the critical exponents of the nonplanar model in one particular case.

If the interaction along the diagonals of the unit cells are all the same and equal to J_2, while the interactions along the sides are different and equal to J_1 and J_1', then the partition function of the nonplanar model in the absence of an external field depends only on the moduli $|J_1 + J_1'|$ and $|J_1 - J_1'|$ and satisfies the relation

$$Z(|J_1+J_1'|,\ |J_1-J_1'|,\ J_2) = Z(|J_1-J_1'|,\ |J_1+J_1'|,\ -J_2). \qquad (1)$$

These symmetry properties can be verified by interchanging rows and columns and changing the directions of the spins in checkerboard order in each second row or column.

The phase diagram of the isotropic nonplanar model ($J_1 = J_1'$), as a function of the temperature and on the relations between the interaction constants, is shown schematically in Fig. 1. In region I there is ferromagnetic ordering (along the diagonals), in region II antiferromagnetic ordering, and region III corresponds to the paramagnetic phase. Point B is determined by the condition that the energies of the ferromagnetic and antiferromagnetic phases be equal at zero temperature ($J_2 = -|J_1|/2$), and at points A and C, i.e., when $|J_1| = 0$, the model breaks up into two identical non-interacting planar Ising sublattices. We are interested in the critical behavior near the point A, when $|J_1| << -J_2$, so that the interaction J_1 between the sublattices can be regarded as a small parameter.

It will be more convenient for us, however, using the symmetry relation (1), to consider an equivalent model with ferromagnetic interaction along the diagonals, and with equal but opposite interactions along the sides. This model is shown in Fig. 2: The dark and light points belong to the first and second sublattices, respectively, the thick lines represent strong ferromagnetic interaction $|J_2|$ and the thin lines, solid and dashed, represent equal but opposite weak interactions $\pm J_1$ between sublattices. Its Hamiltonian is

$$-H/T = K \sum_{(ij)}^{(1)} \sigma_i^{(1)} \sigma_j^{(1)} + K \sum_{(ij)}^{(2)} \sigma_i^{(2)} \sigma_j^{(2)} + \qquad (2)$$

$$+ \lambda \sum_{r} \Delta_1^{(1)}(r) \Delta_2^{(2)}(r) + \lambda \sum_{r'} \Delta_2^{(1)}(r') \Delta_1^{(2)}(r') .$$

Here $K = |J_2|/T$, $\lambda = |J_1|/T$, the superscripts refer to the number of the sublattice, the summation in the first two terms is over the pairs of nearest neighbors of each of the sublattices, and in the last two terms the summation is over the intersections of the diagonals $\vec{r}$ and $\vec{r}'$ (see Fig. 1), which differ in the directions of the sublattice sides. The operators $\Delta_{1,2}$ referred to these intersection points are equal to the differences between neighboring spins in the sublattice (see Fig. 1):

$$\Delta_1^{(1)}(r) = \sigma_1^{(1)} - \sigma_4^{(1)};\ \Delta_2^{(2)}(r) = \sigma_2^{(2)} - \sigma_3^{(2)};\ \Delta_2^{(1)}(r') = \sigma_4^{(1)} - \sigma_5^{(1)};\ \ldots \qquad (3)$$

The Hamiltonian (2) is invariant against the change in the direction of all the spins in one of the sublattices, i.e., it has the same symmetry as the Baxter model, regarded as two Ising sublattices with four-spin interaction between them [3]. At the same time, our model has no symmetry relative to the transition point, analogous to the Kramers-Wannier symmetry of the usual Ising model or to the Sutherland symmetry [5] of the Baxter model, which would make it possible to find the transition curve without calculating the partition function.

We now expand the partition function of the model with Hamiltonian (2) in only even powers of λ. The general term of this series is

$$\frac{\lambda^{2n}}{(2n)!} \sum_{r_1} \ldots \sum_{r_{2n}} \langle \Delta_1(r_1) \ldots \Delta_1(r_{2n}) \rangle \langle \Delta_2(r_1) \ldots \Delta_2(r_{2n}) \rangle + \ldots . \qquad (4)$$

The angle brackets denote averaging over the non-interacting sublattices. The dots denote the terms obtained from the explicitly written terms by making the substitution $\vec{r} \to \vec{r}'$ and corresponding to $\Delta_1 \rightleftarrows \Delta_2$, so that the total number of terms in the curly brackets is q^{2n}.

To estimate the correlators obtained in this manner for the non-interacting sublattices near T_c, when the correlation radius $R_c \sim 1/|\tau| \gg 1$, we use a scaling theory [6, 7], the applicability of which in this case is not subject to doubt. The operators $\Delta_1 = \partial\sigma/\partial x$ and $\Delta_2 = \partial\sigma/\partial y$ have a large critical exponent, equal to 9/8, i.e., they behave under scaling transformations like $R^{-9/8}$, and in particular their pair correlator is $\langle\Delta(r_1)\Delta(r_2)\rangle \sim |\vec{r}_1 - \vec{r}_2|^{-9/4}$, so that the contribution from the summation over large distances ($\sim R_c$) between the operators turns out to be of the order $R_c^{-2-n/2} \sim |\tau|^{2+n/2}$, i.e., negligible.

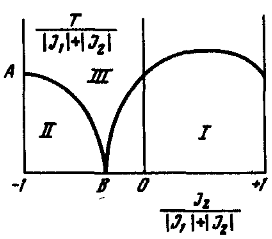

Fig. 1

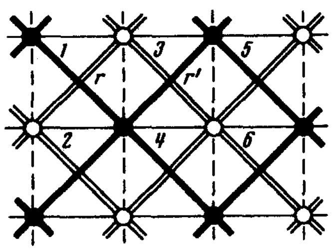

Fig. 2

To the contrary, the main contribution, which is essential for the singular part of the thermodynamic quantities, appears when the operators come pairwise closer, so that the corresponding integrals diverge at short distances, owing to the large critical exponent[1].

According to operator algebra [9, 10], when two operators that enter in the correlator come closer together, reduction takes place. In our case, if two operators Δ (corresponding to the operators $D_{-3/2}$ of [10]) are separated by a distance that is small in comparison with the other distances, then we can make the substitution

$$\Delta_\alpha(r_1)\Delta_\beta(r_2) \to \langle \Delta_\alpha(r_1)\Delta_\beta(r_2)\rangle + \frac{A_{\alpha\beta}}{|r_1 - r_2|^{5/4}}\delta\mathcal{E}\left(\frac{r_1+r_2}{2}\right); \qquad \alpha, \beta = 1, 2; \tag{5}$$

where $\delta\mathcal{E} = \mathcal{E} - \langle\mathcal{E}\rangle$, and $\mathcal{E} = \sigma_i\sigma_{i+1}$ is the energy-density operator while $A_{\alpha\beta} \sim 1$ and depends only on the direction of $(\vec{r}_1 - \vec{r}_2)/|\vec{r}_1 - \vec{r}_2|$. As a result, that contribution of (3) which is important for the thermodynamic singularity is determined by the expression

$$\frac{\lambda^{2n}}{n!}\sum_{r_1}\ldots\sum_{r_n}\langle\mathcal{E}'(r_1)\ldots\mathcal{E}'(r_n)\rangle^2, \tag{6}$$

where $\mathcal{E}' \sim \sigma_i\sigma_j$, and σ_i and σ_j are separated by an effective distance ~ 1.

Expression (6) is analogous to the expression that arises when the partition function of the Baxter model is expanded in terms of the interlattice interaction parameter. The shift of the transition temperature is therefore $T_{c0} - T_c \sim \lambda^2$ and the critical exponent of the specific heat is $\alpha \sim \lambda^2$.

Similarly, expanding the corresponding expressions in powers of λ, and then, with the aid of operator algebra, reducing them to integrals of the "quasi-energy" correlators, we can determine also the remaining critical exponents. The exponent of the correlation radius turns out to be $\nu = 1 - \alpha/2$, that of the magnetization $\beta = (1/8)(1 - \alpha/2)$, and that of the susceptibility $\gamma = (7.4)(1 - \alpha/2)$. The critical exponent of the pair correlation function of the spins does not change in first order in λ^2 ($\eta = 1/4$), and the pair correlator of the energy density behaves at $T = T_c$ like $R^{-2(1-\alpha/2)}$, i.e., the critical exponent of the energy-density operator is $1 - \alpha/2$.

The use of the Kadanoff-Wegner criterion for the variability of the critical exponents together with current algebra confirms these results.

Let us summarize the results. The critical exponents and the transition temperature in the vicinity of the point A on the phase diagram of Fig. 1 vary in proportion to λ^2. The exponents coincide on both sides of the transition point and satisfy the scaling hypothesis. Such a behavior of the exponents should be expected also on the entire transition curve AB, since one can see no reasons why there should appear on this curve a singular point at which the critical behavior of the model can change qualitatively. These properties are apparently connected with the already discussed symmetry of the Hamiltonian (2).

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POLE SINGULARITY OF TRIANGULAR DIAGRAM

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It is well known that the anomalous singularity of triangular diagrams is logarithmic [1 - 3]. If the singularities of the diagram vertices with respect to the virtual masses lie close to the physical region, the character of the anomalous singularity of the entire diagram may change. We shall discuss the case when the singularities of one of the triangular-diagram vertices lie closer to the physical region than the singularities of the two other vertices.

Consider the diagram of Fig. 1. The singularity of the diagram is connected with the region where all three intermediate particles are real. This singularity is close to the physical region if the initial particle 1' forms a weakly bound system of particles 1 and 2. In the particular case when there is no interaction whatever between particles 1 and 2, the block "A" vanishes, particles 1 and 2 are real, and the diagram describes the amplitude of the transition of two particles into three. The "anomalous" singularity of such a diagram is a pole connected with the fact that the particle 3 is real. This pole lies in the physical region, corresponding to arbitrarily large distances between the processes of blocks "B" and "C."

If the binding energy ε of particle I' relative to decay into particles 1 and 2 is much smaller than the masses m_1 and m_2 of these particles, then the two poles of propagators 1 and 2 are close to each other and are on opposite sides of the contour of integration with respect to f_0. Closing this contour around one of these poles, say 2, we obtain the main contribution of the diagram, accurate to terms of order ε/m [2]. This procedure is valid also when $\varepsilon < m_1 \ll m_2$. Putting for simplicity $m_2 \gg m_1, m_3$,[1]), introducing the coupling momentum $\eta = \sqrt{2m_1\varepsilon}$, and taking into account the known relation between the form factor of the vertex "A" and the wave function of the particle 1

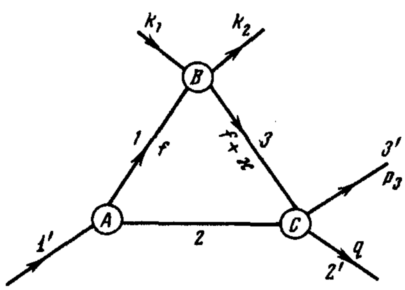

Fig. 1

$$\phi(f) = \frac{F_A(f)}{f^2 + \eta^2}, \qquad (1)$$

[1])The results remain the same also without this condition.