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POLE SINGULARITY OF TRIANGULAR DIAGRAM

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It is well known that the anomalous singularity of triangular diagrams is logarithmic [1 - 3]. If the singularities of the diagram vertices with respect to the virtual masses lie close to the physical region, the character of the anomalous singularity of the entire diagram may change. We shall discuss the case when the singularities of one of the triangular-diagram vertices lie closer to the physical region than the singularities of the two other vertices.

Consider the diagram of Fig. 1. The singularity of the diagram is connected with the region where all three intermediate particles are real. This singularity is close to the physical region if the initial particle 1' forms a weakly bound system of particles 1 and 2. In the particular case when there is no interaction whatever between particles 1 and 2, the block "A" vanishes, particles 1 and 2 are real, and the diagram describes the amplitude of the transition of two particles into three. The "anomalous" singularity of such a diagram is a pole connected with the fact that the particle 3 is real. This pole lies in the physical region, corresponding to arbitrarily large distances between the processes of blocks "B" and "C."

If the binding energy ε of particle I' relative to decay into particles 1 and 2 is much smaller than the masses m_1 and m_2 of these particles, then the two poles of propagators 1 and 2 are close to each other and are on opposite sides of the contour of integration with respect to f_0. Closing this contour around one of these poles, say 2, we obtain the main contribution of the diagram, accurate to terms of order ε/m [2]. This procedure is valid also when $\varepsilon < m_1 \ll m_2$. Putting for simplicity $m_2 \gg m_1, m_3$,[1)] introducing the coupling momentum $\eta = \sqrt{2m_1\varepsilon}$, and taking into account the known relation between the form factor of the vertex "A" and the wave function of the particle 1

$$\phi(f) = \frac{F_A(f)}{f^2 + \eta^2}, \qquad (1)$$

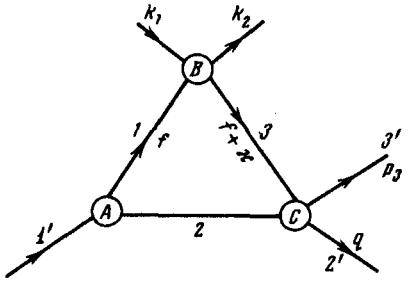

Fig. 1

[1)] The results remain the same also without this condition.

we obtain the following expression for the amplitude of Fig. 1 in the rest system of particle 1':

$$F = F_B F_C M(\kappa, p), \quad M(\kappa, p) = \int \frac{\phi(f)\, d^3 f}{(f + \vec{\kappa})^2 - p^2 - i\epsilon\,(2\pi)^3}, \tag{2}$$

$$\vec{\kappa} = \mathbf{k}_1 - \mathbf{k}_2 = \mathbf{p}_3 + \mathbf{q}, \quad p^2 = (\omega_1 - \omega_2 + m_1)^2 - m_3^2, \quad \omega_i = k_{i0}, \tag{3}$$

where F_B and F_C are the amplitudes of blocks "B" and "C" for the real particles 1, 2, and 3. The momentum p in (3) coincides with the momentum p_3 of particle 3 when $m_2' \gg m_3'$.

We put $p - \kappa = x \ll \kappa + p \simeq 2\kappa$. The approach to the anomalous singularity is due to the decrease of x. We see from (2) that the character of the behavior of M when the singularity is approached is connected with the rate of convergence of the integral

$$\int \phi(f) \frac{d^3 f}{(2\pi)^3} = \psi(0). \tag{4}$$

The wave function $\phi(\vec{f})$ depends, generally speaking, on the radius $a = \eta^{-1}$ of the system and on the interaction radius $\lambda_A^{-1} \equiv \lambda^{-1}$. The significant values of $\bar{f}$, which determine the region of convergence of the integral (4), will therefore be $\bar{f} \sim \lambda, \eta$.

If $x \gg \lambda, \eta$, the denominator in (2) can be taken outside the integral sign, and we obtain $M \simeq -\psi(0)/2\kappa x$, i.e., a power-law growth with increasing x. If $\lambda \gg x \gg \eta$, the wave function in (2) can be replaced by the limiting value at zero interaction radius ($\lambda = \infty$), which is given by formula (1) with $F_A(\bar{f}) = \text{const}$. The integral (2) converges at $2\kappa \gg \bar{f} \gg x$, and we obtain $M \simeq (1/2\kappa)\ln(2\kappa/x)$.

Expression (2) is valid if x and the values of $\bar{f}$ that play an important role in the integral (2) are much smaller than the values of the parameters λ_B and λ_C at which the dependence of the blocks B and C on the virtual masses begins to come into play, i.e., $\lambda_B, \lambda_C \gg x, \lambda, \eta$ [2].

All the foregoing can be illustrated with Hulthen's model:

$$\phi(f) = \frac{4\pi\psi(0)(\mu + \eta)}{(f^2 + \mu^2)(f^2 + \eta^2)}, \quad \mu = \eta + \lambda, \quad \psi(0) = \sqrt{\frac{\mu\eta(\mu + \eta)}{2\pi}}, \tag{5}$$

$$M(\kappa, p) = \frac{\psi(0)}{2\kappa} \frac{i}{\lambda} \left(\ln \frac{\kappa + p + i\eta}{-\kappa + p + i\eta} - \ln \frac{\kappa + p + i\mu}{-\kappa + p + i\mu} \right). \tag{6}$$

As $\lambda \to 0$, Eq. (5) coincides with the wave function of the ground state of the hydrogen atom [4, 5], and

[2] In nuclear reactions at medium energies we have $\lambda_A \sim m_\pi$, whereas $\lambda_B, \lambda_C \sim m_\rho$. At high energies $\omega_1 \gg m$ and at large momentum transfers $\kappa \sim q \sim \omega_1$ the values of λ_B and λ_C in electrodynamic processes are of the order of ω_1; this remains apparently in force also for strong interactions.

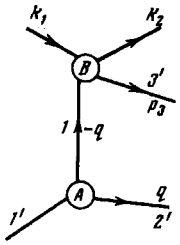

Fig. 2

$$M(\kappa,\ p) \doteq \frac{\psi(0)}{\kappa^2 - (p + i\eta)^2} \approx -\frac{1}{2\kappa}\sqrt{\frac{\eta}{\pi}}\,\frac{\eta}{x + i\eta},\quad x = p - \kappa,\quad \psi(0) = \sqrt{\frac{\eta^3}{\pi}}\cdot \qquad (7)$$

For the states of a hydrogen atom with nonzero orbital angular momentum ℓ the expression for M takes the form of multiple poles [4, 6]. As $\lambda \to \infty$, Eq. (5) is the wave function of a deuteron of zero radius of action, and

$$M(\kappa,\ p) = \frac{i}{2\kappa}\sqrt{\frac{\eta}{\pi}}\ln\frac{\kappa + p + i\eta}{-\kappa + p + i\eta} \approx \frac{i}{2\kappa}\sqrt{\frac{\eta}{\pi}}\ln\frac{2\kappa}{x + i\eta}\cdot \qquad (8)$$

We note that the values of M in both (7) and (8) are of the same order when $|\kappa - p| = x \sim \eta$, while the decrease of (7) with increasing x is much sharper than that of (8). As $\eta \to 0$, the wave function of the hydrogen atom ($\lambda = 0$) goes over into $\phi(0)(2\pi)^3\delta^3(\vec{r})$, the block "A" vanishes, and we obtain the pole (7) in the physical region, whereas at $\lambda \neq 0$ and $\eta \to 0$, the interaction in the block "A" does not vanish, and a logarithmic singularity remains in the physical region (6), (8).

At large momentum transfers $q \gg \eta$ to the particle 2 the diagram of Fig. 1 far from its singularity is in general of the same order as the other diagrams of the process, particularly the pole diagram of the pole approximation of Fig. 2 [7][3], which exists when particles 2 and 3 coincide with 2' and 3' in Fig. 1. At large q, the diagram of Fig. 1 is therefore maximal near the singularity, as is the case, for example in atomic processes [4 - 6], and determines the cross section of the process, which takes the form [6]

$$d\sigma = |2\kappa M(\kappa,\ p)|^2\,\frac{dp}{2\pi}\,d\sigma_B\,d\sigma_C\,, \qquad (9)$$

where $d\sigma_B(\omega_1, \kappa)$ and $d\sigma_C(\kappa, q)$ are the cross sections of the processes described by blocks "B" and "C" with real particles 1, 2, 3. At $\lambda \sim \eta$, the width of the singularity is $dp \sim \eta$, and therefore the value of (9) at $x \sim \eta$ is of the order of $\sigma_B(\sigma_C/a^2)$, where $a = \eta^{-1}$ is the radius of the bound system 1'.

If the nonresonant background is large, it is necessary to add a constant term to the amplitude (2) and an interference term and a nonresonant term to formula (9). The singularity of the diagram can be observed at sufficiently high energies ω_1 and momentum transfers $\kappa \sim q \sim \omega_1$, at which there exists a region $\omega_1 \gtrsim \lambda_B \sim \lambda_C \gg x \gg \lambda, \eta$. To observe the singularity it is necessary to measure the cross section as a function of κ, p, and q.

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[3] The diagram of Fig. 2 is determined by an amplitude equal to $\phi(q)F_B$ and in principal at $q \sim \eta$, by virtue of the pole behavior of the wave function (1), (5). At $\lambda = 0$ (hydrogen), this pole is a multiple one. When $\lambda \neq 0$ this diagram decreases with increasing q like q^{-2} at $\lambda \gg q \gg \eta$ and like q^{-4} at $q \gg \lambda, \eta$.

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PERCOLATION LEVEL IN A THREE-DIMENSIONAL RANDOM POTENTIAL

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The ideas of percolation theory have found extensive use recently in the theory of disordered systems, particularly for the description of the electric conductivity of doped crystalline [1, 2] and amorphous semiconductors [3 - 6]. This posed naturally the mathematical problem of percolation in a continuum, the idea of which is related to the previously investigated lattice problems of "nodes" and "bonds," but does not reduce to the latter. This new problem is formulated as follows. Assume that in all space there is specified an arbitrary random function $V(\vec{r})$ with a finite correlation radius (potential energy). Without loss of generality, we assume that the mean value $\langle V \rangle$ is equal to zero. It is required to determine the so-called percolation level ε_p, i.e., the minimum value of the energy ε at which classically admissible regions, where $\varepsilon > V(\vec{r})$, form paths that go off to macroscopic distances.

It is shown in [1, 2] that in a large number of cases the quantity ε_p determines the activation energy of the conductivity, which can be measured in experiment with sufficient accuracy.

A method of solving continual percolation problems with a computer was first proposed by us in [8]. The same method was verified there with the aid of a two-dimensional problem for which an exact solution is known. We report here for the first time results concerning the three-dimensional problem.

Our purpose was to explain the extent to which the result depends on various properties of the potential. We were interested mainly in a Gaussian potential V defined by the relation

$$V(\mathbf{r}) = \int K(\mathbf{r} - \mathbf{r}') f(\mathbf{r}')\, d^3r', \tag{1}$$

where f is a random Gaussian function with a correlator

$$\langle f(\mathbf{r}) f(\mathbf{r}') \rangle = \delta(\mathbf{r} - \mathbf{r}'), \tag{2}$$

and the kernel $K(\vec{r})$ decreases sufficiently rapidly with r beyond the limits of the correlation radius r_0. The distribution function of the potential is

$$F(V) = \frac{1}{\sqrt{\pi}\gamma} \exp\left(-\frac{V^2}{\gamma^2}\right), \text{where } \gamma^2 = 2\int K^2(\mathbf{r})\, d^3r. \tag{3}$$

We shall refer henceforth not to the percolation energy ε_p, but to the dimensionless quantity ν_c - the fraction of space in which $V < \varepsilon_p$: