SOUND PROPAGATION IN SOLUTIONS OF TWO SUPERFLUID LIQUIDS

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The hydrodynamic equations for solutions of two superfluid liquids were derived in [1], where it was also indicated that undamped sound oscillations of three types should propagate in such solutions. It can be regarded now as established that a phase transition to the superfluid state occurs in liquid He^3 at a temperature on the order of several millikelvin [2]. There can be no doubt that a similar transition can occur in solutions of He^3 in He^4. At sufficiently low temperatures, Cooper pairing of the He^3 atoms should set in and they should go over into the superfluid state. We wish, in this connection, to return to the question of the hydrodynamic properties of a mixture of two superfluid liquids, and investigate, in particular, the propagation of sound in such mixtures.

We write down the hydrodynamic equations of a mixture of two superfluid liquids [1]. The system of equations includes:

a) the continuity equation (ρ is the density and c the concentration)

$$\dot{\rho}_1 + \operatorname{div}(\rho_{s1}\mathbf{v}_{s1} + \rho_{n1}\mathbf{v}_n) = 0; \quad \dot{\rho}_2 + \operatorname{div}(\rho_{s2}\mathbf{v}_{s2} + \rho_{n2}\mathbf{v}_n) = 0 \tag{1}$$

$$\rho_1 = \rho c = \rho_{s1} + \rho_{n1}, \quad \rho_2 = \rho(1-c) = \rho_{s2} + \rho_{n2}$$

($\vec{v}_{s1}$ and $\vec{v}_{s2}$ are the velocities of the superfluid motion of components 1 and 2, respectively, and $\vec{v}_n$ is the velocity of the normal motion[1]);

b) the continuity equation for the entropy

$$\dot{S} + \operatorname{div} S\mathbf{v}_n = 0; \tag{2}$$

c) the equations of the superfluid motions

$$\begin{aligned} \dot{\mathbf{v}}_{s1} + \nabla\left(\mu_1 - \frac{\mathbf{v}_n^2}{2} + \mathbf{v}_n\mathbf{v}_{s1}\right) &= 0, \\ \dot{\mathbf{v}}_{s2} + \nabla\left(\mu_2 - \frac{\mathbf{v}_n^2}{2} + \mathbf{v}_n\mathbf{v}_{s2}\right) &= 0, \end{aligned} \tag{3}$$

where μ_1 and μ_2 are the chemical potentials defined by the following relation for the energy ε:

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[1])The normal density $\rho_{n1} = \rho c - \rho_{s1}$ determines the number of normal atoms of the Fermi component, and differs from the normal density ρ_{nF} of the Fermi excitation by a factor m^*/m, where m^* is the effective mass. The normal density ρ_{n2} therefore includes not only the normal density ρ_{nB} of the Bose excitations (phonons) but also part of the normal density of the Fermi excitations $\rho_{n1}(m^*/m - 1)$, i.e., $\rho_{n2} = \rho_{nB} + \rho_{n1}(m^*/m - 1)$.

$$d\epsilon = TdS + \mu_1 d\rho_1 + \mu_2 d\rho_2 + \rho_{s1}(\mathrm{v}_{s1} - \mathrm{v}_n)\,d(\mathrm{v}_{s1} - \mathrm{v}_n) + \rho_{s2}(\mathrm{v}_{s2} - \mathrm{v}_n)\,d(\mathrm{v}_{s2} - \mathrm{v}_n); \tag{4}$$

d) the equation of total-momentum conservation

$$\mathbf{j} = \rho_{s1}\mathrm{v}_{s1} + \rho_{s2}\mathrm{v}_{s2} + \rho_n \mathrm{v}_n \qquad \dot{j}_i + \partial \Pi_{ik}/\partial x_k = 0, \tag{5}$$

where the momentum flux tensor is equal to ($\rho_n = \rho_{n1} + \rho_{n2}$)

$$\Pi_{ik} = \rho_{s1}\mathrm{v}_{s1i}\mathrm{v}_{s1k} + \rho_{s2}\mathrm{v}_{s2i}\mathrm{v}_{s2k} + \rho_n \mathrm{v}_{ni}\mathrm{v}_{nk} + p\delta_{ik}, \tag{6}$$

and the pressure is

$$p = -\epsilon + TS + \mu_1\rho_1 + \mu_2\rho_2 \tag{7}$$

It is convenient to introduce in place of μ_1 and μ_2 the new potentials $\mu = c\mu_1 + (1 - c)\mu_2$ and $\zeta = \mu_1 - \mu_2$. We then have from (7) the following identity for the pressure ($\sigma = S/\rho$):

$$\frac{1}{\rho}\,dp = \sigma\,dT + d\mu - \zeta\,dc\,. \tag{8}$$

We now obtain the acoustic solutions of the system (1), (2), (3), and (5). Linearizing these equations and eliminating from them the velocities $\vec{\mathrm{v}}_{s1}$, $\vec{\mathrm{v}}_{s2}$, and $\vec{\mathrm{v}}_n$, we obtain three wave equations

$$\begin{gathered}\rho_{s1}(\sigma\Delta T - (1 - c)\Delta\zeta) + \rho\ddot{c} - \rho_{n1}\frac{\ddot{\sigma}}{\sigma} = 0, \\ \rho_{s2}(\sigma\Delta T + c\,\Delta\zeta) - \rho\ddot{c} - \rho_{n2}\frac{\ddot{\sigma}}{\sigma} = 0, \qquad \ddot{\rho} - \Delta p = 0.\end{gathered} \tag{9}$$

We seek solutions such that all the thermodynamic quantities vary like $\exp[i\omega(t - x/u)]$ (plane traveling wave). We choose as the independent variables p, T, and c. Then the condition for the compatibility of the equations in (9) yields a dispersion equation that determines the square of the speed of sound

$$\begin{gathered}u^6 - u^4\left\{\frac{\rho_s}{\rho_n}\bar{\sigma}^2\frac{\partial T}{\partial\sigma} + \frac{\rho_{s1}\rho_{s2}}{\rho_n\rho}\left[\frac{\rho_n}{\rho_s}\right]\frac{\partial\zeta}{\partial c} + \frac{\partial p}{\partial\rho}\left(1 + \frac{\rho_{s1}\rho_{s2}}{\rho_n\rho}\left[\frac{\rho_n}{\rho_s}\right]\frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial c}\right)^2\right)\right\} + \\ + u^2\left\{\left(\frac{\rho_s}{\rho_n}\bar{\sigma}^2\frac{\partial T}{\partial\sigma} + \frac{\rho_{s1}\rho_{s2}}{\rho_n\rho}\left[\frac{\rho_n}{\rho_s}\right]\frac{\partial\zeta}{\partial c}\right)\frac{\partial p}{\partial\rho} + \frac{\rho_{s1}\rho_{s2}}{\rho_n\rho}\sigma^2\frac{\partial T}{\partial\sigma}\left(\frac{\partial\zeta}{\partial c} + \frac{\partial p}{\partial\rho}\frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial c}\right)^2\right)\right\} - \\ - \frac{\rho_{s1}\rho_{s2}}{\rho_n\rho}\frac{\partial\zeta}{\partial c}\sigma^2\frac{\partial T}{\partial\sigma}\frac{\partial p}{\partial\rho} = 0.\end{gathered} \tag{10}$$

We have introduced here, for brevity, the notation

$$\left[\frac{\rho_n}{\rho_s}\right] = c\frac{\rho_{n1}}{\rho_{s1}} + (1 - c)\frac{\rho_{n2}}{\rho_{s2}},$$

$$\frac{\bar{\sigma}^2}{\sigma^2} = 1 + \frac{2}{\rho_s}(\rho_{s1}(1 - c) - \rho_{s2}c)\frac{1}{\sigma}\frac{\partial\sigma}{\partial c} + \frac{\rho_{s1}\rho_{s2}}{\rho_n\rho}\left[\frac{\rho_n}{\rho_s}\right]\left(\frac{1}{\sigma}\frac{\partial\sigma}{\partial c}\right)^2.$$

In the derivation of (10) we have neglected the terms with the derivatives $(\partial \rho/\partial T)$. Allowance for these terms results in negligibly small errors in the sound velocities. The obtained equation is cubic in the square of the speed of sound u^2. It has three real roots corresponding to three types of undamped sound waves. We note that if one of the mixture components is not superfluid ($\rho_{s1} = 0$), then the first equation in (9) loses its wave character, and establishes only a connection between the oscillations of the concentration c and the entropy σ. In this case the dispersion equation becomes quadratic and determines the speeds of the known first and second sounds in solutions of He^3 in He^4 [3].

The roots of (10) can be easily obtained because one of the roots is small in comparison with the two others. The first root of this equation

$$u_1^2 = \frac{\partial p}{\partial \rho} \tag{11}$$

determines the speed of first sound, at which the compression (pressure) waves propagate. The second root

$$u_2^2 = \frac{\rho_s}{\rho_n} \bar{\sigma}^2 \frac{\partial T}{\partial \sigma} + \frac{\rho_{s1}\rho_{s2}}{\rho_n \rho} \frac{\partial \zeta}{\partial c} \left[\frac{\rho_n}{\rho_s}\right] \tag{12}$$

determines the speed of "second" sound, at which the temperature and concentration oscillations propagate.

Finally, the third root

$$u_3^2 = \frac{\rho_{s1}\rho_{s2}\sigma^2}{\rho_s \rho \bar{\sigma}^2} \frac{\partial \zeta}{\partial c} \Bigg/ \left(1 + \frac{\rho_{s1}\rho_{s2}}{\rho_s \rho \bar{\sigma}^2}\left[\frac{\rho_n}{\rho_s}\right] \frac{\partial \zeta}{\partial c} \frac{\partial \sigma}{\partial T}\right) \tag{13}$$

determines the speed of "third" sound[2]. The propagation of the "third" sound is a unique property of a mixture of two superfluid liquids. At the phase transition point ($\rho_{s1} = 0$) the velocity u_3 vanishes, and the speed of second sound is equal to

$$u_2^2 = \frac{\rho_s}{\rho_n}\left(\bar{\sigma}^2 \frac{\partial T}{\partial \sigma} + c^2 \frac{\partial \zeta}{\partial c}\right) \tag{14}$$

(which coincides with the result obtained in [3] for a solution of nonsuperfluid He^3 in He^4).

The density ρ_{n1} should decrease exponentially with decreasing temperature. Near $T = 0$, the speed of third sound tends to the value

$$u_3^2 = c(1 - c)\frac{\partial \zeta}{\partial c} . \tag{15}$$

The speed of second sound u_2 tends in this limit ($\rho_{n1} \ll \rho_{n2}$) to the value

$$u_2 = u_{10}/\sqrt{3} \tag{16}$$

[2]This sound must not be confused with that propagating over films of liquid helium II, which is also called "third."

(u_{10} is the speed of first sound at $T = 0$).

In (11), (12), and (13), we have neglected small terms of relative order $c[(1/\rho)(\partial\rho/\partial c)]^2$, which for weak solutions is always legitimate ($c < 0.06$ in degenerate solutions of He^3 in He^4).

Propagating with the speed of "third" sound are the oscillations of the concentration c. An analysis of (9) shows that these oscillations are coupled with the oscillations of the density ρ and of the temperature T; this makes it easy to excite the latter oscillations.

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[2] T.A. Alvesalo, Yu.D. Anufriev, H.K. Collan, O.V. Lounasmaa, and P. Wennerstrom, Phys. Lett. A43, 175 (1973).
[3] I.M. Khalatnikov, Zh. Eksp. Teor. Fiz. 29, 265 (1955) [Sov. Phys.-JETP 2, 98 (1956)]; Zh. Eksp. Teor. Fiz. 23, 169 (1952).