Self-similar adiabatic motions of a self-gravitating gas in a star

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New solutions of the problem of explosion of a star are obtained. All the asymptotic forms of the self-similar accretion of the gas to the center are derived. Exact solutions with a converging shock wave are indicated.

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Self-similar spherically symmetrical adiabatic motions of an ideal self-gravitating gas were first considered in[1,2] as a model of supernova explosions (a complete bibliography can be found in[3]). The system of hydrodynamic equations for the self-similar solutions reduces in the presence of a mass-conservation integral[2] to an autonomous system of three differential equations for functions of the self-similar variable $\ln\lambda$, where $\lambda = r(\beta A f t^2)^{-1/w}$. The parameters of the problem are the adiabatic exponent $\gamma > 1$ and $1 < w < 3$.

A detailed investigation of this system by the methods of qualitative theory of differential equations leads to the following new results:

1. At $\gamma < 4/3$ and $w > 5/2$ all the solutions of the problem of the stellar explosion, at shock-wave Mach numbers $M \sim 1$, tend as $\lambda \to 0$ $(t \to \infty)$ to an equilibrium state

$$\rho = A r^{-w}, \quad M = 4\pi A r^{3-w}/(3-w), \quad p = 2\pi A^2 f r^{2(1-w)}/(w-1)(3-w), \; v = 0 \quad (1)$$

(the notation is standard). Thus, after the explosion, the star again returns to a state similar to its initial one. These solutions can be used to simulate repeated explosions of stars. We note that for these solutions, at $\gamma < 4/[3+(2w-5)^2/8(w-1)]$, the gas velocity executes as $\lambda \to 0$ an infinite number of oscillations about zero, so that the approach of the gas to the equilibrium state is oscillatory. A similar regime of gas motion after passage of a shock wave was obtained earlier in an analogous problem in general relativity theory.[4]

At $\gamma < 4/[3+(2w-5)^2/8(w-1)]$ and $10/7 < w < 5/2$ the stellar-explosion problem has a denumerable set of solutions corresponding to the infinite sequence M_i $(i = 1,2,\ldots)$ of the shock-wave Mach numbers: $M_i \to 1$ as $i \to \infty$. As $\lambda \to 0$ these solutions take the asymptotic form

$$M = C_1 \beta A r^{3-w} \lambda^{2a(w-1)}, \quad \rho = C_1 \beta A \frac{3\gamma a}{4\pi} \lambda^{2a(w-1)} r^{-w},$$

$$p = C_2(\beta A f)^{2/w} f^{-1} t^{-4(w-1)/w}, \quad v = \frac{4(w-1)}{3\gamma w}\frac{r}{t}, \tag{2}$$

$$a = (3 - w)/(3\gamma - 2(w-1)).$$

All the solutions of the problem of stellar explosion for $M_i < M < M_{i+1}$ have an expanding cavity inside the gas. In these solutions, the gas velocity v after the passage of the shock wave has with decreasing λ a certain finite number of oscillations $N(M)$, with $N(M) \to \infty$ as $M \to 1$. The exact "dynamic explosion of equilibrium" solution indicated in[2] at $\gamma = 7/6$ and $w = 12/5$ belongs to the described class of solutions with asymptotic form (2); for this solution we have $M^2 = 15/2$.

Self-similar solutions with formation of a cavity inside of a gas at $w = 5/2$ were investigated in detail in[2]. An analysis of the system in the general case shows that there are no solutions with cavity formation at $\gamma < 2(w-1)/3$, nor do solutions of the problem of stellar explosion exist in this range of parameters.

2. Self-similar solutions that describe accretion of the gas to the center can have as $\lambda \to 0$ only one of the following two stable asymptotic forms, at $4/3 < \gamma < 5/3$

$$M \sim t^{2(3-w)/w}, \quad \rho \sim r^{-1/(\gamma-1)} t^{2(-1 + 1/w(\gamma-1))},$$

$$p \sim r^{-\gamma/(\gamma-1)} t^{-2(2-(3\gamma-2)/w(\gamma-1))},$$

$$-v \sim r^{(3-2\gamma)/(\gamma-1)} t^{-1-2(4-3\gamma)/w(\gamma-1)} \tag{3}$$

and at $1 < \gamma < 5/3$

$$M \sim t^{2(3-w)/w}, \quad \rho \sim r^{-3/2} t^{(3-2w)/w},$$

$$p \sim r^{-3\gamma/2} t^{-4+2(2+3\gamma/2)/w}, \quad -v \sim r^{-1/2} t^{(3-w)/w}. \tag{4}$$

In these asymptotic solutions, a pointlike mass that increases with time is produced at the symmetry center at $t \neq 0$, so that in the classical theory (3) and (4) describe gas accretion in a "black hole." In the asymptotic solutions (3) and (4) the pressure, density, and temperature of the gas are infinite, the gas falls into the center within a finite time, with the gas-flow Mach number $M \to 0$ in the asymptotic form (3) and with $M \to \infty$ as $\lambda \to 0$ in the form (4).

At $\gamma = 5/3$ there is a stable asymptotic form, similar to (3) and (4), in which the Mach number of the gas flow tends to an arbitrary constant. At $\gamma = 4/3$ there is, in place of (3), another stable asymptotic form of accretion, wherein the gas falls into the center in an infinite time and no pointlike mass is produced. There is no self-similar accretion at $\gamma > 5/3$.

3. The self-similar solutions with converging shock waves have been determined ıt $t>0$ and $r>0$ (see[5]). The following exact solutions satisfies at $\nu=(2\gamma-1)/2(\gamma-1)$ all the conditions for matching together on the shock wave, At $)<\gamma<\gamma_1$ we have

$$M = \frac{r^3}{2ft^2} V^2, \quad v = \frac{r}{(-t)} V, \quad \rho = \frac{(3-w)V^2}{ft^2 4\pi(2+wV)} \quad p = 0, \tag{5}$$

vhere the function $V(\lambda)>0$ is defined by the equation

$$V^{-1}(1+3V/2)^{1-w/3} = \lambda^{w/2}, \tag{6}$$

nd the parameter λ_1 is calculated from (6) at $V=4/w(\gamma-1)$. At $\lambda>\lambda_1$ the solution is he equilibrium state (1).

The solution (5) and (6) in the region ahead of the shock wave describes a flux of ustlike matter traveling from the center. As $r\to 0$, this solution has the asymptotic ›rm (3) (subject to the substitutions $t\to -t$ and $v\to -v$)—the analog of a "white ole" in classical theory. At all γ and w there exist solutions analogous to (5) and (6), ı which the pressure p ahead of the shock wave is greater than zero (if $p=0$, then a ›lution can be obtained in analytic form for all γ and w). These solutions indicate the :gime of contraction of the "white hole" by the shock wave, wherein the gas goes over ıto the equilibrium state.

We note in conclusion that the formulation of the problem of stellar explosions[2] ; well as some of the results of the present paper hold also in general relativity theory. he equilibrium distribution (1) is then replaced by the static solution of Einstein's ıuations with an equation of state $p=k\epsilon$ $(0\leqslant k<1)$, where $\epsilon\sim r^{-2}$:

$$ds^2 = r^{4k/(1+k)} c^2 dt^2 - \frac{1+6k+k^2}{(1+k)^2} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

hich can be transformed, by a special change of coordinates, into a self-similar uation.

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