# Nonlocal hydrodynamics of phonon gas in dielectrics

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It is shown that the character of the hydrodynamic phenomena in dielectrics is substantially altered. If the dispersion of the transverse oscillations has certain properties leading to a sharp increase of the relaxation time of the longitudinal long-wave phonons (LLP). The results explain the experimentally observed temperature dependence of the thermal-conductivity and second-sound-damping coefficients.

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In pure dielectrics at low temperatures, the mean free path $l_N$ of the thermal phonons is determined by the normal decay and coalescence processes that are due to ternary anharmonicities. Under the influence of only normal collisions, the distribution that is established in the phonon system is

$$\left(\exp\frac{\omega-\mathbf{up}}{T+\delta T}-1\right)^{-1}\approx N(\omega)-\left(\mathbf{up}-\frac{\omega}{T}\delta T\right)\frac{\partial N}{\partial\omega}, \tag{1}$$

and is characterized by two hydrodynamic parameters: the ordered-motion velocity $\mathbf{u}(\mathbf{r},t)$ and the nonequilibrium temperature increment $\delta T(\mathbf{r},t)$. Here $\omega$ is the phonon energy, $\mathbf{p}$ is its quasimomentum, and $N(\omega)$ is the Bose distribution function.

The purpose of the present paper is the following. It is usually assumed that hydrodynamic phenomena in dielectrics are determined exclusively by thermal phonons (see, e.g.,[1,2]). It is shown here that when certain conditions, which are realized in crystals of both high and low symmetry, are satisfied, the situation is substantially altered: the carriers of the energy and of the momentum are as a rule the thermal phonons, whereas the viscosity is determined by the longitudinal long-wave phonons (LLP).

It is known that the LLP can participate in collisions of two types: (a) for the processes $l+t\rightleftarrows l$ and $t+t\rightleftarrows l$ ($l$ and $t$ are the longitudinal and trasverse modes) we have asymptotically at $\omega\ll T$ a mean free path $\tau(\omega)\sim T^{-1}\omega^{-4}$; (b) for the process $l+t\rightleftarrows t$, which is possible in the presence of degeneracy of the transverse oscillations (Herring[3]), we have $\tau(\omega)\sim T^{m-5}\omega^{-m}$, with $m=2$, 3, or 4, depending on the crystal-lattice symmetry. This shows already that in a number of low-symmetry crystals, for which $m=3$ and 4, the viscosity is determined by the LLP: according to (1), the momentum density of these phonons is $\propto\omega^2$, whereas the distance over which the momentum is transferred is $\propto\omega^{-m}$ This means that phonon-gas layers moving with different velocities $\mathbf{u}$ will interact with one another via LLP whose mean free path is comparable with the characteristic dimensions of the system. It is clear that under

  

these conditions the hydrodynamic equations have a nonlocal character. However, most of the experimentally investigated crystals have a high symmetry, with $m=2$, and the contribution of the LLP might seem to be negligible.

We note in this connection that in high-symmetry crystals the maximum distance $\Delta p$ between the constant-energy surfaces of the transverse oscillations is usually small. Under these conditions Herring's process has a threshold at the frequency $\omega_0 \approx \Delta p/pT$, since a longitudinal phonon with momentum larger than $\Delta p$ "cannot be fitted" between the temperature-dependent transverse oscillation modes. Thus, at $\omega_0 < \omega < T$ only processes (a) are possible. If the mean free path $l(\omega_0) \approx l_N (p/\Delta p)^4$ exceeds the characteristic dimensions of the system, then the case $m=4$ is realized (It is assumed that the LLP mean free path connected with the Simons mechanism (see[4]) exceeds the characteristic dimensions of the system). In the general case the hydrodynamic equations of the phonon gas, which are nonlocal in space and time, are of cumbersome form. We therefore consider by way of example stationary heat transport in a plate of thickness $2d \gg l_N$. We direct the $z$ axis normal to the surface of the plate and the $x$ axis along the temperature gradient and the drift velocity. Under conditions of diffuse scattering of the phonons by the plate boundaries[1)], the velocity $u(z)$ satisfies the equation

$$\int_{-d}^{+d} Q(|z-z'|)\,[u(z)-u(z')]\,dz' + u(z)\,[K(d+z)+K(d-z)] = \gamma \frac{\partial T}{\partial x},$$

$$K(z) = \int_{-1}^{1} d\mu\,(1-\mu^2) \int_0^\infty \frac{d\omega}{\tau(\omega)} \frac{\partial N}{\partial \omega} \left(\frac{\omega}{T}\right)^4 \exp\left(-\frac{z}{l(\omega)|\mu|}\right),$$

$$Q = -dK/dz, \quad \gamma \approx s^2 T^{-1}, \quad l(\omega) = s\tau(\omega), \quad \omega = sp, \quad \mu = s_z/s. \tag{2}$$

The integral term in Eq. (2) describes the viscous-friction force, while the terms $u(z)K(d \pm z)$ correspond to interaction with the boundaries. The expressions for these forces can be easily obtained from fairly obvious considerations. The probability of emission of a nonequilibrium LLP with momentum $p$ in an element $dz$ is proportional to $dz|l(\omega)\mu|^{-1}u(z)p_x \partial N/\partial\omega$, and the probability of absorption of this phonon in the element $dz'$ is proportional to $dz'|l(\omega)\mu|^{-1}\exp(-|z-z'|/|l(\omega)\mu|^{-1})$. In addition, account must be taken of the momentum of flux density $p_x s_z$, as well as the inverse process, wherein the element $dz'$ emits a phonon which is absorbed by the element $dz$. Summing the contributions from the exchange of all the possible phonons, we arrive at the expression $Q(|z-z'|)\times[u(z)-u(z')]dzdz'$. The force of interaction with the boundary can be determined by a similar reasoning, except that the probability of phonon absorption by the boundary is equal to unity and the boundary does not emit nonequilibrium phonons.

Omitting the straightforward calculations, we write down the result for the thermal-conductivity coefficient $\kappa$ in the form ($C$ is the specific heat)

$$\kappa = Cs \frac{d^2}{l_N^{eff}}, \qquad l_N^{eff} \approx \begin{cases} (dl_N^3)^{1/4} & m=4 \\ l_N \ln d/l_N, & m=3 \\ l_N & m=2 \end{cases} \tag{3}$$

We present also an expression for the second-sound damping coefficient

$$\operatorname{Im}\Omega / \Omega \approx \Omega l_N^{eff}(\lambda) / V.$$

Here $\Omega$ is the frequency, $V$ is the velocity of the second sound, and $l_N^{\text{eff}}(\lambda)$ is determined from formulas (3) in which $d$ is replaced by the second-sound wavelength $\lambda = 2\pi V \Omega^{-1}$. (The result is understandable: the LLP whose mean free path exceeds $\kappa$ make no noticable contribution to the damping.)

We note that the available experimental data do not agree with the predictions of the local hydrodynamic theory ($m=2$). According to the latter $l_N^{\text{eff}} = l_N \sim T^{-5}$, whereas a reduction of the experimental results by the formula $l_N^{\text{eff}} \sim T^{-\alpha}$ yields in all cases $3 < \alpha < 4$. Hydrodynamic phenomena have been observed by now in four substances, $^4$He, $^3$He, Bi and NaF. However, reliable data concerning the temperature dependence are available only for $^4$He and NaF.[2] According to Herring's classification, these substances correspond to $m=2$. However, $\Delta p/p \lesssim 1/5$, for $^4$He and $\Delta p/p \lesssim 1/7$, for NaF, so that Herring's process has a threshold. the exponent $15/4 = 3.75$, obtained in our analysis at $m=4$, is quite close to the experimentally observed results: $\alpha \approx 3.71$ for NaF and $\alpha = 3$–$4$ for $^4$He.

A more detailed exposition of the questions dealt with above will be published in the journal "Fizika nizkikh temperatur" [Soviet Journal of Low Temperature Physics].

[1)]Even in reflection from an ideal surface, the LLP have a unity probability of being transformed into transverse phonons, and these can easily transfer the momentum to the thermal phonons.[5]


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