Optimal quantum measurements in detectors of gravitation radiation

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When continuous measurement is made with an instrument whose interaction with the gravitational antenna has a Hamiltonian proportional to the instantaneous coordinate, the sensitivity of the detector has in principle a limit. It becomes necessary to use a procedure with a pulsed measurement of the coordinate.

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A second generation of gravitation detectors of the Weber type, which have a sensitivity sufficient for the observation of bursts of gravitational radiation from hypothetical astrophysical sources located outside our galaxy, are being presently con-

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structed in ten laboratories of several countries. The antennas employed are deeply cooled massive aluminum cylinders or relatively light dielectric single-crystal cylinders and it is proposed to convert the mechanical oscillations into electrical ones either with capacitive sensors or with devices such as SQUID.[1-4] The sensitivity of the gravitational-wave detectors is determined by the relaxation processes in the antenna itself and by the fluctuations in the measurement circuit. If the relaxation time of the antenna oscillations is long enough and the antenna temperature is low, then the decisive role is played by the measurement errors. Under these conditions the acceleration sensitivity of a detector using the aforementioned sensors operating in the usual continuous regime is equal, in the quantum limit[1-5] to

$$\left(\frac{F}{m}\right)_{\min} \approx \frac{2}{\tau}\sqrt{\frac{\hbar\omega}{m}}, \tag{1}$$

$\hbar$ is Planck's constant; ω and m are the natural frequency and the equivalent mass of the antenna; τ is the time of action of a train of waves of frequency $\approx\omega$).

On the other hand, it is known that if the sensor were to be able to determine the change of the antenna energy with accuracy $\hbar\omega$ (the possibility of such a measurement is admissible in quantum mechanics[6]), then the sensitivity would be[7]

$$\left(\frac{F}{m}\right)_{\min} \approx \frac{2}{\tau}\sqrt{\frac{\hbar\omega}{mn}}; \tag{2}$$

where $n \gg 1$ is the initial value of the quantum number. In this case, in contrast to (1), the sensitivity increases formally without limit with increasing n.

The purpose of this article is to identify the cause of the sensitivity limit in the former case and to formulate conditions for the realization of an optimal quantum meter in a gravitation detector.

The observation of the action exerted on the oscillator consists of comparing the values of the amplitude (energy), phase, or instantaneous coordinate before and after the action. The measurement sensitivity is higher the more accurate the measurement of one of the foregoing quantities and the more accurately it is reproduced after repeated measurements in the absence of action. The measurement accuracy is determined not only by the fluctuations in the measuring instrument, but also by the type of interaction between the measuring apparatus and the oscillator.[8] If, for example, the purpose is to measure the vibrational energy of the oscillator, and the Hamiltonian of the interaction with the measuring apparatus is proportional to the instantaneous coordinate (as is the case for a capacitive pickup and a SQUID), then the error in the measurement of the energy is at best $\sqrt{n}$ quanta.[6] This result is a direct consequence of the uncertainty relation

$$\Delta x\, \Delta p \geqslant \frac{\hbar}{2}. \tag{3}$$

The sensors used in gravitational detectors can be used for accurate measurement of the instantaneous coordinate. However, if they work continuously, then the error in the measurement of the runnning value of the coordinate will have a definite limit.

Indeed, measurement of the coordinate is accompanied by perturbation of the momentum, which increases the uncertainty of the coordinate in the succeeding instants of time. The minimal error $\Delta x_{\min}$ of a continuous measurement of the running value of the coordinate can be determined by starting from the known miminal error of the energy measurement. The quantity $\Delta x_{\min}$ is simultaneously the error in the determination of the oscillation amplitude. Consequently, the following condition should be satisfied

$$m\omega^2 \Delta x_{\min} \sqrt{2n\hbar\omega/m\omega^2} = \sqrt{n}\hbar\omega, \tag{4}$$

i.e.,

$$\Delta x_{\min} = \sqrt{\hbar/2m\omega}\ . \tag{5}$$

Relation (5) determines the minimum measurement error of the running value of the coordinate and of the oscillator amplitude in the case of continuous measurement with an instrument whose interaction Hamiltonian with the oscillator is proportional to the instantaneous coordinate.

Under the influence of the force $F\cos(\omega t+\phi)$, the average energy of an oscillator in a state with an indeterminate phase changes by an amount $\epsilon_F=(F\tau)^2/8m$, and the energy uncertainty is increased by[9]

$$\Delta\epsilon_F = \sqrt{2n\hbar\omega\epsilon_F}, \tag{6}$$

which is equivalent to an rms change of the oscillation amplitude

$$\delta x_F = \frac{F\tau}{2\sqrt{2}m\omega}\ . \tag{7}$$

From the condition $\delta x_F \geqslant \Delta x_{\min}$ or $\Delta\epsilon_F \geqslant \sqrt{n}\hbar\omega$, we obtain relation (1). The limitation (1) on the sensitivity is the consequence of the non-optimal method used for the measurements. To attain a sensitivity exceeding the limit (1) it is necessary either to search for a sensor whose interaction Hamiltonian with the antenna is diagonal in the energy representation, or to use known sensors in a different mode.

The harmonic oscillator has the following important property. Its wave packet repeats with a period $2\pi/\omega$. An external force changes the average (over the ensemble) value of the instantaneous coordinate at the instants $t_k=2\pi k/\omega$ (k is an integer) by an amount

$$\delta x_F = (F\tau/2m\omega)\sin\phi. \tag{8}$$

In formula (8) the phase is reckoned from the instant of the first measurement of the coordinate. The sensitivity of the method is determined by the accuracy of the measurement of the initial ($t=0$) and final $t=t_f$ value of the coordinate. Consider the case when one uses for the preparation of the initial state and for the measurement of the final coordinate one and the same procedure of measurement of the coordinate averaged over the measurement time $t_m \ll \omega^{-1}$. The minimal coordinate-measurement error is determined by two opposing tendencies. With increasing measurement time t_m, on the one hand, the measurement accuracy increases, and on the other hand the spreading of the wave packet increases. The momentum perturbation increases the width of the wave packet, at $t_m \ll \omega^{-1}$, by an amount $\Delta p t_m/2m = \hbar t_m/4m\Delta x_r$ (Δx_r is the width of the packet determined by its reduction in the measurement). The error in the determination of the average coordinate is

$$\Delta x = \sqrt{(\Delta x_r)^2 + (\hbar t_m/4m\Delta x_r)^2}\,. \tag{9}$$

At $\Delta x_r = \sqrt{\hbar t_m/4m}$ the value of Δx is a minimum[10]

$$\Delta x_{\min} = \sqrt{\hbar t_m/m}\,. \tag{9a}$$

Accordingly, the error in the measurement of the coordinate difference at the instant $t=0$ and at $t=t_f$ is

$$\sqrt{2}\Delta x_{\min} = \left(\frac{\hbar t_m}{m}\right)^{1/2}. \tag{10}$$

From (8) and (10) we get

$$\left(\frac{F}{m}\right)_{\min} \approx \frac{2}{r}\sqrt{\frac{\hbar\omega}{m}\,\omega t_m}\,. \tag{11}$$

The quantity ωt_m can be arbitrarily small, and therefore the sensitivity of this method is formally limited only by relativistic effects. Relation (9a) is valid at $\Delta x_{\min} \gg \hbar/mc$ (c is the speed of light).[11] Formula (11), strictly speaking, pertains to a harmonic oscillator with one degree of freedom. The antennas in gravitational detectors have a large number of natural frequencies, which can be excited in the case of pulsed measurements. However, if the frequencies are equidistant, then the periodicity of the repetition of the mean value and of the uncertainty of the coordinate of the endpoint of the antenna is conserved. It must be emphasized that in continuous measurements the oscillator amplitude is perturbed by an amount $(\hbar/2m\omega)^{1/2}$, and in the case of pulsed (strobing) measurements it is perturbed by $(\hbar/2m\omega^2 t_m)^{1/2}$. That is to say, strobing measurements causes a substantial "heating" of the antenna. This effect can be one of the technical reasons of the sensitivity limit. A sensitivity corresponding to formula (11) can be attained at $\sin\phi \simeq 1$. Since the phase of the gravitational pulse is not determined, to preserve a high sensitivity level we must use two gravitational detectors whose measurements are shifted by $\pi/2\omega$ in time.

Analysis has shown that the fundamental premises described above remain in force also in the case when account is taken of the thermal fluctuations in the sensor and in the amplifier. In the case of a capacitive sensor, we must replace $\hbar$ in the relations above by $2\kappa T_{\rm eff}/\Omega$, where Ω is the operating frequency of the sensor, $\kappa T_{\rm eff} = \hbar\Omega/2 + \hbar\Omega/[\exp(\hbar\Omega/\kappa T) - 1]$, and κT is the energy of the thermal oscillations in the sensor. The noise temperature of the amplifier was also assumed equal to $T_{\rm eff}$.

Summarizing the foregoing, we emphasize that the use of ordinary sensors in periodic measurement of the instantaneous coordinate makes it possible to circumvent the sensitivity limit (1) which is typical of the continuous-measurement regime.

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