Canonical formalism for field theories with singular Lagrangians

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A canonical formalism using the Routhian is proposed, in contrast to the traditional formalism in which the Hamilton function is used. The non-canonical coordinates are chosen to be the field components for which in the Hamilton theory the conjugate momenta are coupled. The Routh equations are obtained for the field as well as the analog of the Schrödinger equation for the state vector.

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It is known that in the canonical formulation of any problem in mechanics, only some of the entire set of coordinates can be regarded as canonical.[1] In this case one introduces in place of the Hamiltonian the Routhian function, which is a Hamiltonian function with respect to the canonical coordinates and a Lagrangian function with respect to the remaining coordinates. The choice of canonical and non-canonical coordinates is determined by the character of the problem. In field theory one customarily uses the Hamiltonian, but it is more natural to use the Routhian in the case of field theories with singular Lagrangians, by considering only "nondegenerate" field components as the canonical coordinates. The "degenerate" momenta do not appear in this

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case at all, and consequently, the constraints on these momenta (of the family and those derived from them) likewise do not arise.

We consider a field $\Phi^N = (\phi^N, \psi^N)$ with a Lagrangian $\mathcal{L}$ such that the definition of the momentum conjugate to Φ^N does not lead to any constraints at $\Phi^N = \phi^N$ ("nondegenerate" components), and yields a primary constraint at $\Phi^N = \psi^N$ ("degenerate" components). We shall regard only the ϕ^N as canonical coordinates. Their conjugate momenta are

$$\pi_N = \partial \mathcal{L} / \partial \dot{\phi}^N . \quad (1)$$

For $\mathcal{R} \equiv \pi_N \dot{\phi}^N - \mathcal{L}$ we have[1)]

$$\delta \int \mathcal{R}\, d^4x = \int [\, \dot{\phi}^N \delta \pi_N - \left(\frac{\partial \mathcal{L}}{\partial \phi^N} - \frac{\partial}{\partial x^a} \frac{\partial \mathcal{L}}{\partial \phi^N_{,a}} \right) \delta \phi^N \quad (2)$$

$$- \left(\frac{\partial \mathcal{L}}{\partial \psi^N} - \frac{\partial}{\partial x^k} \frac{\partial \mathcal{L}}{\partial \psi^N_{,k}} \right) \delta \psi^N \,]\, d^4x \quad ;$$

expressing $\mathcal{R}$ the Routh-function density) in terms of π_N, ϕ^N, ψ^N and *regarding these variables as independent*, we find that the system

$$\dot{\phi}^N = \delta R / \delta \pi_N \,, \qquad \dot{\pi}_N = - \delta R / \delta \phi^N , \quad (3a)$$

$$(\delta / \delta \psi^N) \int R\, dt = 0 \quad (3b)$$

(the Routhian is $R \equiv \int \mathcal{R} d^3x$) coincides with the Lagrange equations for Φ^N, with (3a) having an explicit canonical form (the Routhian R is a Hamiltonian for ϕ^N and π_N and is a Lagrangian for ψ^N); Hamilton's principle can be rewritten in the form

$$\delta \int (\pi_N \dot{\phi}^N - \mathcal{R})\, d^4x = 0. \quad (3c)$$

By virtue of (3) we have for an quantity A

$$dA / dt = \partial^* A / \partial t + \{ R, A \} , \quad (4)$$

where $\partial^* A / \partial t$ is taken at constant ϕ^N and π_N, and the Poisson brackets are

$$\{ A, B \} \equiv \int d^3x [(\delta A / \delta \pi_N)(\delta B / \delta \phi^N) - (\delta B / \delta \pi_N)(\delta A / \delta \phi^N)] . \quad (5)$$

In the quantization we assume $\{A, B\} \to i[A, B]$, so that for ϕ^N, π_N, ψ^N the only nonvanishing commutator is

$$[\pi_N (\mathbf{x}, t), \phi^M (\mathbf{x}', t)] = - i\, \delta^M_N\, \delta (\mathbf{x} - \mathbf{x}') \quad (6)$$

(the degenerate components are consequently not quantized).

We emphasize that by virtue of the independence of the variables the question of compatibility of (6) with the constraint equations or the sequence in which they are expanded does not arise.[2] Of course, if ϕ^N and π_N are regarded as the solution of the equations of motion then, since they are consequently not independent, it is possible for them not to satisfy the canonical relations (6). For example in electrodynamics Eq. (6) is not compatible with the equation $\mathrm{div}\mathbf{E}=0$, but there is no contradiction here, inasmuch as in our formalism this equation is not a constraint. In classical mechanics this circumstance can be seen even with free motion of a particle as an example.

We note in conclusion that since there is no Hamiltonian in this formalism, the Schrödinger equation cannot be formulated. As can be seen from the correspondence principle, it should be replaced by the equation

$$(i\partial^* /\partial t - R)| > = 0, \tag{7}$$

where $\partial^*/\partial t$ has the same meaning as in (4), i.e., it takes the time dependence into account both explicitly and via the non-quantized components ψ^N.

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[1]It is assumed that $c=\hbar=1$, $i,k=0,1,2,3$, $\alpha,\beta=1,2,3$, and that Einstein's summation rule is used.

[1]L.D. Landau and E.M. Lifshitz, Mekhanika (Mechanics), Nauka, 1965, p. 168 [Pergamon, 1968]. H. Goldstein, Classical Mechanics, Addision-Wesley, 1950 [Russian transl. Nauka, 1973, p. 243].
[2]P.A.M. Dirac, Principles of Quantum Mechanics, Oxford, 1968 [Russ. trans., Mir 1968, p. 16].