One-dimensional collapse of plasma waves

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It is shown, with upper-hybrid quasipotential plasma waves as an example, that one-dimensional collapse of oscillations takes place in systems described by a Schrödinger equation with nonlocal nonlinearity.

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This paper deals with the singularities of the self-action of quasimonochromatic oscillations in one-dimensional physical systems described by a modified nonlinear Schrödinger equation of the type

$$-i\frac{\partial e}{\partial \tau}+\frac{\partial^2 e}{\partial \xi^2}+\alpha e|e|^2-\beta e\frac{\partial^2|e|^2}{\partial \xi^2}=0. \qquad (1)$$

This equation differs from the well known Schrödinger equation with local nonlinearity ($\beta=0$) in the presence of one-dimensional collapse—an effect wherein the analytic solutions are "destroyed" by the onset of a singularity after a finite time.

Equation (1) is valid, in particular, for packets of Langmuir oscillations in an isotropic plasma, which move with supersonic velocity for different types of high-frequency quasipotential waves (hybrid and cyclotron) propagating in a magnetized low-pressure plasma perpendicular to the constant magnetic field, and also for excitons in one-dimensional lattices (the latter is shown in[1]).

We illustrate the procedure of obtaining Eq. (1) using as an example upper-hybrid quasipotential oscillations. We represent the equation for the complex amplitude of the electric field of the wave

$$E_{\rm hf}=\tfrac{1}{2}[E(x,t)\exp(i\omega t-ikx)+\text{c.c.}]$$

in a coordinate system connected with the group velocity u, in the form

$$-2i\omega_0\frac{\partial E}{\partial t}+\gamma v_{Te}^2\frac{\partial^2 E}{\partial \xi^2}-(\omega_{pe}^2 n+2\omega_{He}^2 h)E=0. \qquad (2)$$

Here n and h are slow (in the scale of $2\pi/\omega_0$) relative perturbations of the electron density and of the magnetic field, and are produced under the influence of the Miller forces; ω_{pe} and ω_{He} are the Langmuir and electron cyclotron frequencies, respectively;

$$\omega_0=(\omega_{pe}^2+\omega_{He}^2)^{1/2},\ \omega-\omega_0\ll\omega_0,$$

$$\xi=x-ut,\ u=\frac{\gamma v_{Te}^2 k}{\omega},\quad \gamma=\frac{3\omega_{pe}^2}{\omega_{pe}^2-3\omega_{He}^2}.$$

Equation (2) is considered in the region of the normal dispersion of the oscillations ($\omega_{pe}^2 > 3\omega_{He}^2$) with the condition $L_E^2 \gg r_{He}^2$ (r_{He} is the cyclotron radius of the electrons) satisfied by the spatial scale L_E of the field amplitude.

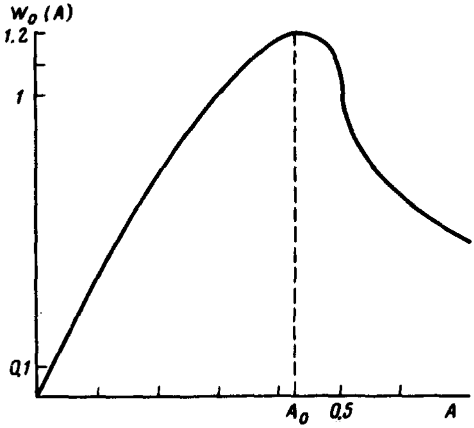

FIG. 1.

Their dependence of small perturbations ($n \ll 1$, $h \ll 1$) on the amplitude E can be described by the linearized system of equations of two-fluid hydrodynamics of a collisionless plasma, supplemented by the Miller force. If $u^2 \ll v_A^2$, $\tau_E^2 \omega_{He}\omega_{Hi} \gg 1$ (τ_E is the time scale in the group-velocity frame), then it is easily found that

$$n = -\frac{\psi}{v_A^2} + L_0^2 \frac{\partial^2}{\partial \xi^2}\left(\frac{\psi}{v_A^2}\right), \qquad h = -\frac{\psi}{v_A^2}, \tag{3}$$

where

$$v_A^2 = \frac{H_0^2}{4\pi M N_0}, \quad L_0 = \frac{c}{\omega_{pe}}, \quad \psi = \frac{|E|^2}{16\pi M N_0}.$$

In dimensionless variables, (2) and (3) reduce to Eq. (1) with $\alpha = 1$ and $\beta = 1$.

We consider first stationary solutions of (1) in the form $e = \epsilon(\xi)\exp(-iA^2\tau)$, using as an example the case $\alpha = 1$, b$=1$.[1)] It is easy to show that for the integral curves on the (ϵ, $d\epsilon/d\xi$) phase plane we obtain the equation

$$\left(\frac{d\epsilon}{d\xi}\right)^2 = \frac{A^2\epsilon^2(1-\epsilon^2/2A^2) + C}{1-2\epsilon^2}, \tag{4}$$

where C is the integration constant. For isolated solutions (solitons), which correspond to $C = 0$, we can write the analytic expression

$$\pm A\xi = \frac{1}{2}\ln\frac{\sqrt{1-\epsilon^2/2A^2} + \sqrt{1-2\epsilon^2}}{|\sqrt{1-\epsilon^2/2A^2} - \sqrt{1-2\epsilon^2}|} - 2A\ln\frac{\sqrt{1-2\epsilon^2} + \sqrt{2}\sqrt{2A^2-\epsilon^2}}{\sqrt{|1-4A^2|}} \tag{5}$$

Analysis of the integral curves shows that at $A \geqslant 1/2$ the derivative of the solitons has a singularity at $\xi = 0$ ("sharply pointed" solitons).[2] At $A \ll 1/2$ the contribution of the nonlocal nonlinearity can be neglected:

$$\epsilon(\xi) = \sqrt{2}\,A\ \mathrm{ch}^{-1}(A\xi). \tag{6}$$

At $A \gg 1/2$ we have for the "sharply pointed" solitons[2]

$$\pm A\xi = \mathrm{arch}\left(\frac{1}{\sqrt{2}\epsilon}\right) - \sqrt{1 - 2\epsilon^2}. \tag{7}$$

Solitary analytic solutions satisfy a known criterion,[3] according to which these solutions are stable if the condition

$$\frac{dW_0}{dA^2} > 0 \tag{8}$$

is satisfied, where $W_0 = \int \epsilon^2(\xi)\, d\xi$ is the total energy of the soliton. The expression for $W_0(A)$ is of the form

$$W_0(A) = \frac{1}{2}(1 - 4A^2)\ln\frac{1 + 2A}{|1 - 2A|} + 2A. \tag{9}$$

According to the plot of $W_0(A)$ shown in Fig. 1, there exists a limiting value of the amplitude $\sqrt{2}\,A_0$ ($A_0 \approx 0.415$), and this value separates the stable solutions $A < A_0$ from the unstable analytic solutions. The "sharply pointed" solitons should also be unstable.[3]

In the investigation of the dynamics of arbitrary initial distributions we can use the integrals of Eq. (1)—the conservation laws for the number of quanta and for the Hamiltonian of the system:

$$I_1 = \int |e|^2 d\xi, \tag{10}$$

$$I_2 = \int \left[\left|\frac{\partial e}{\partial \xi}\right|^2 - \frac{\alpha}{2}|e|^4 - \frac{\beta}{2}\left(\frac{\partial |e|^2}{\partial \xi}\right)^2\right] d\xi. \tag{11}$$

From (1), (10), and (11) follows the integral relation:

$$\frac{d^2}{d\tau^2}\int \xi^2 |e|^2 d\xi = 8I_2 + 2\int \left[\alpha |e|^4 - \beta\left(\frac{\partial |e|^2}{\partial \xi}\right)^2\right] d\xi. \tag{12}$$

For $\beta = 1$ and $\alpha = -1$ the evolution of any distribution with $I_2 < 0$ leads to the onset of a singularity due to the sharpening of the $e(\xi)$ profile. At $\alpha = 1$, the local nonlinearity hinders the formation of singularities of the solution. But if the term with the local nonlinearity is negligible in the initial distribution, then it becomes possible to construct self-similar "collapsing" solutions.

To this end, we represent e in the form $e=a\exp(-i\phi)$ and consider the system of equations for the real amplitude and the real phase:

$$\frac{\partial a^2}{\partial \tau}+\frac{\partial}{\partial \xi}\left(2a^2\frac{\partial \phi}{\partial \xi}\right)=0, \qquad (13)$$

$$\frac{\partial \phi}{\partial \tau}+\left(\frac{\partial \phi}{\partial \xi}\right)^2=\frac{1}{a}\frac{\partial^2 a}{\partial \xi^2}+a^2-\frac{\partial^2 a^2}{\partial \xi^2}. \qquad (14)$$

We seek the solution of (13) and (14) in the geometrical-optics approximation, i.e., neglecting the term $[1/a(\partial^2 a/\partial \xi^2)]$, and without taking into account the locally nonlinear term a^2. The resultant system has self-similar solutions that take for $v=a^2$ and $u=\partial\phi/\partial\xi$ the form:

$$U=(\tau_0-\tau)^{-3/5}U_0(\eta), \quad V=(\tau_0-\tau)^{-2/5}V_0(\eta), \qquad \eta=\frac{\xi}{(\tau_0-\tau)^{2/5}}. \qquad (15)$$

The corresponding approximation of the self-similar distribution is a fourth-order parabola:

$$V_0=\begin{cases}(\eta_0^2-\bar{\eta}^2)^2, & \bar{\eta}\leq\eta_0;\\ 0, & \bar{\eta}>\eta_0;\end{cases} \qquad \bar{\eta}=\frac{\eta}{(200)^{1/4}}. \qquad (16)$$

It is important that the obtained self-similar solution conserves the total energy in the "collapse," and the terms discarded in (14), including the diffraction term $(\partial^2 a/\partial\xi^2)$, remain small up to the instant $\tau=\tau_0$ of formation of the singularity. It is obvious that in a real system the "collapse" should lead to the appearance of dissipation.

Thus, collapse of one-dimensional distributions of high-frequency plasma oscillations is possible under the action of nonlocal nonlinearity. This conclusion is valid, in particular, for packets of Langmuir oscillations in an isotropic plasma, which move with supersonic velocity, and for which the onset of nonlocal nonlinearity ($\alpha=-1$, $\beta=1$) is connected with violation of the quasineutrality of the perturbations of the density of the electrons and ions under the influence of the ponderomotive force.

1) Stationary solutions of the system (2), (3) were considered in[2] for the particular case of a strongly nonlocal nonlinearity $L_E \ll L_0$.

2) It is obvious that this singularity does not appear when account is taken of the viscosity or of the nonlinear dissipation.

3) It is impossible to prove (8) rigorously for solutions with singularities, but it can be regarded as the asymptotic limit of the solution as $A\to 1/2$.

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