Three-dimensional solitons in He II

S. V. Iordanskiĭ and A. V. Smirnov

L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences
(Submitted 10 April 1978)
Pis'ma Zh. Eksp. Teor. Fiz. **27**, No. 10, 569–572 (20 May 1978)

It is shown that three-dimensional stationary wave packets—solitons—can propagate in He II in the case of positive dispersion. A numerical solution of the equation that describes these packets is obtained, and the corresponding branch of the excitation spectrum is calculated. The question of excitation of a soliton by an electromagnetic wave is considered.

PACS numbers: 67.40.Pm

According to the experimental data on sound damping,[1,2] the phonon part of the He II spectrum has positive dispersion at low pressures.

The nonlinearity of the medium may stop the spreading of the waves under the influence of the dispersion. In the one-dimensional case, the nonlinear equations of the hydrodynamics of He II at $T=0$ reduce, at any sign of the dispersion, to the Korteweg—de Vries equation. Therefore one-dimensional solitons always exist in He II. We shall show below that at positive dispersion there can exist in the He II three-dimensional axially-symmetrical solitons that propagate with a velocity lower than that of sound. The opposite case is impossible because of the phonon emission.

We confine ourselves henceforth to the hydrodynamic approximation, and consider the case $T=0$. The hydrodynamic equations can be written in Hamiltonian form if we choose the canonical variables to be the density ρ and the velocity potential ϕ. Taking into account the cubic anharmonicity and dispersion, we write down the Hamiltonian in the form

$$H = \int d^3r \left\{ (\rho_0 - \rho')\frac{(\nabla\phi)^2}{2} + \frac{1}{2}\frac{c_s^2}{\rho_0}\rho'^2 + \frac{\gamma c_s^2}{\rho_0}(\nabla\rho')^2 + \frac{\beta c_s^2}{\rho_0^2}\frac{\rho'^3}{6} \right\}, \tag{1}$$

where ρ_0 is the equilibrium density, ρ' is the deviation of the density from the equilibrium value, c_s is the speed of sound, and $\beta=[2\partial \ln c_s/\partial \ln\rho - 1]$. This Hamiltonian leads to the dispersion law $\omega = c_s k(1+\gamma k^2)$.

We seek the solution in the form of an axially-symmetrical packet propagating along the x axis, so that the x-component of the velocity of the liquid takes the form

$$u_x = \partial\phi/\partial x = -2b[\beta+3]^{-1}c_s f(b, \xi, \eta), \tag{2}$$

$$\xi = b^{1/2}(2\gamma)^{-1/2}(x - vt), \quad \eta = b(2\gamma)^{-1/2} r_\perp, \quad v = c_s\left(1 - \frac{b}{2}\right). \tag{3}$$

In the lowest order in $b \ll 1$, which corresponds to soliton dimensions much larger than atomic, the hydrodynamic equations reduce to the equation

$$\frac{\partial^4 f}{\partial \xi^4} - \frac{\partial^2 f}{\partial \xi^2} - \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial f}{\partial \eta}\right) = -\frac{\partial^2 f^2}{\partial \xi^2}. \tag{4}$$

(This equation corresponds to the Kadomtsev–Petviashvili equation.) Thus, the soliton comprises a perturbation with a characteristic transverse dimension $\sqrt{2\gamma/b}$ much larger than the longitudinal $\sqrt{2\gamma}/\sqrt{b'}$.

Equation (4) was solved numerically by a method proposed by Petviashvili for an analogous two-dimensional equation,[3] the only difference being that a difference equation with respect to the variable η was solved instead of a Fourier transformation. The isolines of the function f, which is even in the variable ξ, are shown in Fig. 1.

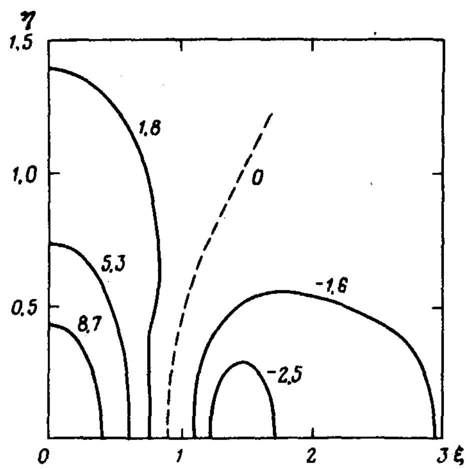

FIG. 1. Isolines of the function $f(\xi,\eta)$. At the origin, f has a maximum equal to 12.2, and at the points ($\eta=0,\xi=\pm 1.4$) it has a minimum equal to -3.1.

The existence of such a solution determines an additional branch in the spectrum of the excitations of He II. Knowledge of the function f makes it possible to calculate the energy and the momentum of the soliton as functions of its velocity. The expression for the soliton momentum

$$\mathscr{P} = \mathscr{P}_x = \int d^3 r\,(\rho_0 + \rho')\,u_x \tag{5}$$

diverges formally, since the function f has an asymptotic form of dipole character at infinity. This divergence can be eliminated by assuming that the soliton accelerates slowly and begins to increase at an instant of time t_0. At the instant t, at the distances $R > c_s(t-t_0)$ from the start of the acceleration, the velocity of the liquid is determined by the initial form of the soliton. Since the integral of the linear term in $\mathscr{P}$ reduces to an integral over the surface, this term makes no contribution to the change of the momentum.

To determine the soliton energy E as a function of the momentum, it is necessary to eliminate the soliton velocity, using the relation $v=\partial E/\partial \mathscr{P}$. The dispersion law at large momenta, accurate to terms proportional to $1/\mathscr{P}^2$, is of the form

$$E(\mathscr{P}) = c_s \mathscr{P} + \frac{2^6 \rho_0^2 c_s^3 \gamma^3}{\mathscr{P}(\beta+3)^4} I^2 + \text{const}, \quad I = 2\pi \int\int f^2(\xi, \eta) d\xi \eta d\eta. \quad (6)$$

Numerical integration yields a value $I=231$. Expression (6) contains the integration constant, which cannot be calculated theoretically. One can only expect its value not to exceed, in order of magnitude, the characteristic atomic energy. Figure 2 shows a plot of $E(\mathscr{P})$ with a value zero for the constant. The values $c_s=2.383\times 10^4$ cm/sec and $\beta=4.68$ were taken from[4], from which the value $\gamma=8\times 10^{-17}$ cm^2 at zero pressure was also taken.

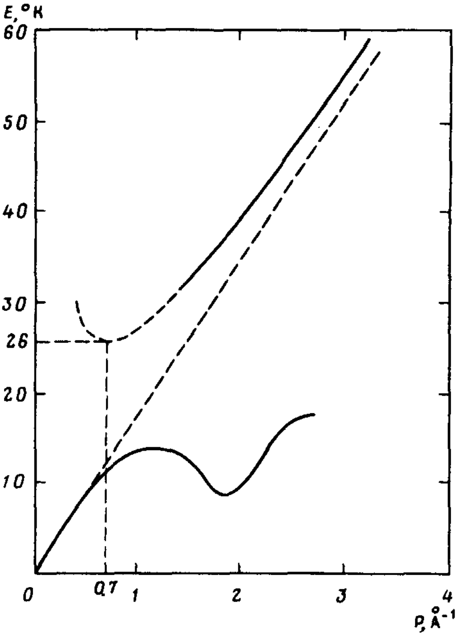

FIG. 2. Soliton spectrum continued arbitrarily into the region where the hydrodynamic approximation is violated. The spectrum of single-particle excitations of the He II is shown for the sake of clarity.

The soliton is a multiparticle excitation of large dimensions with a large lifetime. We consider the possibility of exciting a soliton by an electromagnetic wave, confining ourselves to the case of light incident along the x axis, and assume that the wavelength of the light is larger than the longitudinal dimension of the soliton. The excitation mechanism consists of scattering by the inhomogeneity brought about the the soliton. The angular dependence of the scattering amplitude can be calculated in the Born approximation, and the dipole singularity at large distances causes the main contribution to the transport cross section to be made by backscattering at angles $\pi-\theta \lesssim b^{1/2}$.

Using the form of the singularity f at large distances, we can calculate the main contribution to the transport cross section.

Assuming that the parameters of the soliton vary slowly and that it preserves its form, we obtain, equating the momentum lost by the light in scattering per unit time to the change of the soliton momentum

$$\frac{db}{dt} = - E_0^2\left(\rho_0 \frac{\partial\epsilon}{\partial\rho_0}\right)^2 \frac{(2\gamma)^{3/2} l\,|\mathbf{q}|^4}{(4\pi)^2 \rho_0 c_s \epsilon} b^{-1/2}, \tag{7}$$

where E_0 is the electric-field amplitude, $\mathbf{q}$ is the wave vector of the incident wave of light, and ϵ is the permittivity. The equation (7) remains valid until the soliton longitudinal dimension becomes comparable with the light wavelength.

In the transparency region we have $\rho_0(\partial\epsilon/\partial\rho) \approx 0.05$, and consequently at $|\mathbf{q}| = 10^5$ cm^{-1} and at a radiation intensity 10^6 W/cm^2 the coefficient of $b^{-1/2}$ in the right-hand side of (7) is of the order of 10^{-6} sec^{-1}. Thus, an attempt to excite solitons with light in the transparency region would seem ineffective. It should be noted that when the frequency of the light is increased the excitation conditions should improve because of the increase of ϵ.

It is possible that more favorable conditions for the excitation of such solitons exists in other inert gases, since it appears that the superfluid properties of He II are not very significant. At the present time, however, we know the sign of γ only for He II.

[1]B.M. Abraham *et al.*, Phys. Rev. A **1**, 250 (1970).
[2]R.P. Roach, Phys. Rev. Lett. **25**, 1002 (1970).
[3]V.I. Petviashvili, Fiz. Plazmy **2**, 469 (1976) [Sov. J. Plasma Phys. **2**, 257 (1976)].
[4]H.J. Maris, Phys. Rev. Lett. **28**, 277 (1972).