Leptons and quarks in the quaternion model

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A gauge theory is constructed for quaternion fields of spin 0, 1/2, and 1. It is shown that this theory is equivalent to the usual theory with spontaneously broken $[SU(2)\times U(1)]_{\rm loc}\times SU(2)_{\rm glob}$ symmetry and to the isodoublet structure of the multiplets of material fields—fermions (quarks and leptons) and Higgs scalars; the GIM mechanism is realized in the theory in a natural fashion and CP-invariance is violated.

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We wish to call attention in this article to the possibility of describing in the quaternion model (hereafter the Q-model) the systematics of leptons and quarks, and of regarding the gauge version of this model as a unified theory of weak and electromagnetic interactions[1)].

We consider the Q-model of a free fermion field

$$L_\psi = -\frac{1}{2}\,\overline{\psi}\gamma\partial\psi - \frac{1}{2}m\overline{\psi} + Q.c\ , \tag{1}$$

where the field ψ, by assumption, is the quaternion $\psi=\psi_0+e_k\psi_k$, $e_k(k=1,2,3)$— three imaginary Q units with a multiplication table $e_ie_j=-\delta_{ij}+\epsilon_{ijk}e_k$, $e_k^{Q.c}=-e_k$ ["$Q.c$" denotes the quaternion conjugation and "$+$" denotes the total (quaternion and Hermitian) conjugation; $\overline{\psi}=\psi^+\gamma_4$]. The mass parameter in the Lagrangian (1) is also a quaternion; it is assumed here that $m^+=m$. The states corresponding to the fields ψ have an analogous form $\psi=\psi_0+e_k\psi_k$, where the components $\psi_{0,k}$ are in the general case complex functions of the space-time coordinates. For the norm of such a state to be a real positive number, it suffices (as can be easily verified directly by using the rule of quaternion multiplication) to assume $\psi_{0,k}=|\psi_{0,k}|e^{i\delta}$, i.e., $\delta_0=\delta_1=\delta_2=\delta_3=\delta$ (the condition that the quaternion is quasi-real). It is easily seen that in this case the composition relation which is obiligatory in quaternion mechanics[1,2] is also satisfied—if $\psi=\psi_1\psi_2$, then the norm $N(\psi)=N(\psi_1)N(\psi_2)$.

The Lagrangian L_ψ is invariant to global transformations

$$\psi\rightarrow q\psi\ , \tag{2a}$$

$$\psi \to e^{i\theta}\psi , \tag{2b}$$

where q is a normalized quaternion, $q=\exp[e_k\theta_k]$. The invariance to the transformations (2b) is a consequence of the fact that the quaternion of the state ψ is quasi-real. We proceed now to local transformations, i.e., let the phases θ and $\theta_k (k=1,2,3)$ depend on the 4-coordinates. Then, following the standard prescription, we construct a covariant derivative of the form

$$D_\mu \psi = \partial_\mu \psi - g C_\mu \psi - g' B_\mu \psi , \; C_\mu = i e_k C^k_\mu \tag{3}$$

with the transformation law (2). This leads directly to the laws for the transformation of the gauge fields C_μ and B_μ

$$C'_\mu = q C_\mu q^{-1} + \frac{1}{g}\partial_\mu q q^{-1}, \; B'_\mu = B_\mu + \frac{i}{g'}\partial_\mu \theta . \tag{4}$$

The gauge-invariant Lagrangian of the fields ψ, C_μ, and B_μ is of the form

$$L = -\frac{1}{2}\bar{\psi}(\gamma_\mu D_\mu \psi - \frac{1}{2} m)\bar{\psi}\psi - \frac{1}{8}(\partial_\mu C_\nu - \partial_\nu C_\mu + g[C_\mu , C_\nu])^2$$

$$- \frac{1}{8}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + Q.c. . \tag{5}$$

We now write down in L the fields μ and C_μ and the parameter m component by component (it is convenient to assume here $\psi=\psi_0+ie_k\psi_k, m=m_0+ie_km_k$) and introduce the Tiomno matrices[3] $F^{(\pm)i}$ with elements $(i,j,k=1,2,3)$

$$F^{(\pm)i}_{00} = 0 , \; F^{(\pm)i}_{0k} = F^{(\pm)i}_{k0} = \mp \delta_{ik} , \; F^{(\pm)i}_{jk} = -i\epsilon_{ijk} , \tag{6a}$$

$$F^{(\pm)i} F^{(\pm)j} = \delta_{ij} + i\epsilon_{ijk} F^{(\pm)k} , \; [F^{(+)i}, F^{(-)j}] = 0. \tag{6b}$$

By a unitary transformation[3] of the matrix $F^{(\pm)i}$ together with the fields

$$\hat{\psi} = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

into the matrices [relations (6b) guarantee the existence of such a unitary transformation T; σ^i and Pauli matrices]

$$\eta^i = \sigma^i \times 1 = T F^{(+)i} T^{-1}, \; \tau^i = 1 \times \sigma^i = T F^{(-)i} T^{-1}, \; [\eta^i , \tau^j] = 0 \tag{7}$$

("×" denotes an outer product) and to the fields $S=T\hat{\psi}$, we have

$$L = -\bar{S}\gamma_\mu[\partial_\mu - i g C^i_\mu \tau^i - i g' B_\mu]S - m_0 \bar{S}S + m_i S \eta^i S - \frac{1}{4}(C^i_{\mu\nu} + g C^j_\mu C^k_\nu \epsilon_{ijk})^2 - \frac{1}{4} B^2_{\mu\nu}. \tag{8}$$

We carry out a symmetry analysis. The group of the invariant transformations of the Lagrangian (8) is given by the tranformations

$$S \to \exp[i \frac{\tau^k}{2} \theta^k]S, \quad \tau^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \tag{9a}$$

$$S \to \exp[i\theta] \tag{9b}$$

(we have made use of $[\eta^i, \tau^j] = 0$). It is this which fixes the transformation properties of the field S. Obviously, under the transformations (9a) the two upper and two lower components in the 4-column S becomes "intermixed," i.e., S behaves as an aggregate of two independent isospinars S_1 and S_2

$$S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}.$$

In the absence of the term $m_i \bar{S}\eta^i S$ the symmetry of the Lagrangian increases from $[SU(2)\times U(1)]_{\text{loc}}$ [see (9)] to $[SU(2)\times U(1)]_{\text{loc}}\times SU(2)_{\text{glob}}$($SU(2)_{\text{glob}}$ "intermixes" the doublets (S_1 and S_2), and moreover, isotopic degeneracy sets in. The isospin properties of the fields are undetermined: two doublets–singlet and triplet. The mass term lifts the degeneracy.

For the scalar Q-fields with standard Lagrangian—including a mass term and a self-action term with respect to parameters

$$m^2 = (m^2)_0 + i e_k (m^2)_k, \; h = h_0 + i e_k h_k \,/\, (m^2)^+ = m^2, \; h^+ = h$$

all the statements made concerning the field ψ remain in force: a) the transformation law (2)[2]; b) the form of the covariant derivative (3); c) the isodoublet structure of the fields D_1 and D_2, where

$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} \equiv D = T\hat{\phi}, \quad \hat{\phi} = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}.$$

A detailed analysis of the Higgs potential[4] in the Lagrangian of the fields D_1 and D_2

$$P_D = (m^2)_0 D^+D + (m^2)_k D^+\eta^k D + h_0 [(D^+D)^2 + (D^+\eta^k D)^2] + 2h_k (D^+D)(D^+\eta^k D) \tag{10}$$

reveals that the vacuum in the theory is unstable and the fields D_1 and D_2 develop the vacuum expectation values

$$< D_1 > = \begin{pmatrix} 0 \\ \lambda_1 \end{pmatrix}, \quad < D_2 > = \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix} e^{i\epsilon}$$

(the presence of the relative phase shift ϵ is due to the complex character of the mass matrix of the D field). As a result we obtain a theory with spontaneously broken $SU(2)\times U(1)$ gauge symmetry with vector mass fields

$$\frac{C^1_\mu \pm i\, C^2_\mu}{\sqrt{2}}, \quad g\, C^3_\mu + g' B_\mu / \sqrt{g^2 + g'^2}),$$

massless field

$$-g' C^3_\mu + g\, B_\mu / \sqrt{g^2 + g'^2}),$$

and a quartet of fermions (the number of these quartets is in the general case arbitrary) in the form of a pair of doublets. Furthermore, after application of the Higgs procedure,[4] five massive scalar bosons remain in the theory.

Any subsequent concretization of the obtained theory now depends only on the spatial properties of the weak interaction. If we assume the transformation laws (2) only for a left-spiral field ψ_L, and assume for the right-spiral field ψ_R only the simple phase transformation (2b), then we arrive at a model of the Weinberg–Salam type[4] for two doublets of particles, for example the leptons

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix} \text{ and } \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}.$$

The mass term of the field ψ drops out in this case of the Lagrangian (5) and the lepton masses stem from the invariant relation

$$G(\bar{\psi}_L \phi \psi_R + \bar{\psi}_R \phi^+ \psi_L] + Q.c., \quad G = G_0 + i e_k G_k (G^+ = G) \qquad (11)$$

as a result of the vacuum shift of the scalar field ϕ. The inclusion of the quark quartet u, d, s, and c in the model is connected with the introduction of still another quarternion ψ' in the thoery[1)]. The presence of two quark doublets in ψ' guarantees a natural realization of the Glashow-Illiopoulis-Maiani (GIM) mechanism[4] for all types of mixing of quarks due to the relation (11). This relation breaks the $SU(1)_{\rm glob}$ symmetry and serves as a basis for the calculation of the Cabibbo angle; moreover, by virtue of the presence of a nontrivial relative phase in the vacuum expectation values of the fields D_1 and D_2, it leads to a superweak violation of the CP invariance on account of exchange of Higgs bosons. We propose to consider these important applications separately.

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[1)]A number of attempts to use quarternions to obtain an isotopic classification of mesons and baryons were made in the early sixties,[1-3] especially in connection with the development of quarternion quantum mechanics.[1,2]

[2]A "vector" law of transformation $\phi \rightarrow q\phi q^{-1}$ would lead to the absence of a quarternion mass parameter for the field ϕ.

[3]The new heavy leptons and quarks are located, according to our model, in additional quartets (induced by the quarternions ϕ and ϕ', respectively) with doublet-doublet structure as before.

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