High orders of perturbation theory in the Yang-Mills model with scalar field

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(Submitted 28 March 1978)
Pis'ma Zh. Eksp. Teor. Fiz. **27**, No. 10, 594–599 (20 May 1978)

In the model of a scalar field interacting with Yang-Mills fields, we investigate field configurations that make the largest contribution to the higher order of perturbation theory in the scalar-field interaction constant λ and in the Yang-Mills constant g. The contributions of these configurations to the expansion of the Green's functions is estimated.

PACS numbers: 11.10.–z

A method developed in[1] was recently used to obtain the expansion coefficients of Green's functions in high orders of perturbation theory for scalar models of field theory,[1,2] scalar electrodynamics,[3,4] the Yang-mills theory,[5] and in fermion models.[6] To use this method it is necessary to choose for the classical equations a solution that makes the largest contribution to the functional integral. In the case of scalar models[1,2] it was shown in[3], by using the Sobolev inequality,[7] that this property is possessed by a spherically-symmetrical solution. In[4] it was proposed to find the form of the solution for a system of interacting fields by a method that ensures optimality of the solution in the region where the theory differs insignificantly from the case of a pure scalar field.

We apply here this method to the theory of Yang-Mills A_μ^a interacting with an isodoublet of scalar mesons ϕ. The Euclidean action for this model is of the form

$$S = \int L d^4x, \quad L = \frac{1}{4}(F_{\mu\nu}^a)^2 + |(\partial_\mu - \frac{ig}{2}\tau^a A_\mu^a)\phi|^2 + \frac{\lambda}{2}|\phi|^4. \tag{1}$$

The mass term has been left out of (1), since we consider Green's functions at short distances.

In accordance with[1], the saddle-point values for the fields $\tilde{A}_\mu^a$ and $\tilde{\phi}$ and for the constants $\tilde{g}$ and $\tilde{\lambda}$ in the functional integral for the expansion coefficients $C_n^{(k,m)}$ of the Green's functions

$$G_n = \sum_{km} \lambda^k g^{2m} G_n^{(k,m)} \tag{2}$$

are determined from the condition that the functional J be extremal:

$$\delta J = 0, \quad J = S + m \ln g^2 + k \ln \lambda . \tag{3}$$

We choose a solution of the classical equations with center at zero and with unity scale. It is convenient in this case to map the four-dimensional space on the surface of a unit sphere in five-dimensional space[8]:

$$z_\mu = \frac{2x_\mu}{x^2+1}, \quad z_5 = \frac{x^2-1}{x^2+1}, \quad dS_5 = \left(\frac{2}{1+x^2}\right)^4 d^4x, \quad A_\mu^a = \frac{\partial z_i}{\partial x_\mu} A_i^a(z),$$
$$\phi(x) = \frac{2}{1+x^2} Y(z). \tag{4}$$

We can impose on the five-component vector $A_i^a(z)$ two additional conditions:

$$Z_i A_i^a = 0, \quad (\partial_i - z_i(z\partial)) A_i^a = 0; \tag{5}$$

the second condition fixes the gauge. Substitution of (4) in (1) reduces the action to a form invariant to the 10-parameter group of rotations of a sphere.

In accordance with the method of[4], we begin the search for the solutions (3) with the case $m \ll k$. In this case $|A_i^a| \ll Y$, so that we can use as the zeroth approximation the solution for the theory of a pure scalar field[1,2]

$$\tilde{Y}^{(0)} = \left(-\frac{2}{\lambda}\right)^{1/2} u, \tag{6a}$$

where u is a constant isospinor ($u^+u=1$). The equation for A_i^a can in this case be linearized. Just as in[4], in first-order approximation we have for A_i^a a solution in the form of superposition of first harmonics on a sphere:

$$\tilde{A}_i^b = \frac{a}{g} \eta_{ik}^b z_k, \qquad a \sim m/k \ll 1. \tag{6b}$$

The constant matrices η_{ik}^b, by virtue of the condition that the equations for the next higher approximations in a have solutions, satisfy the relations:

$$\eta_{ik}^a = -\eta_{ki}^a; \quad [\eta^a, \eta^b] = C_1 \epsilon_{abc} \eta^c, \quad (\eta^a)^2 \eta^b = C_2 \eta^b,$$
$$\eta^a Tr(\eta^a \eta^b) = C_3 \eta^b, \tag{7}$$

where C_1, C_2, and C_3 are arbitary constants. There are six types of such matrices; they can be chosen in the form

$$\text{I.}\ \eta^a_{ik} = 2\epsilon_{abc}(\gamma_i)^{bd}(\gamma_k)^{cd} \qquad \text{IV.}\ \eta^1_{ik} = R_{ik},\ \eta^2_{ik} = S_{ik}$$

$$\text{II.}\ \eta^a_{ik} = \epsilon_{aik} \qquad \text{V.}\ \eta^1_{ik} = R_{ik} + S_{ik}$$

$$\text{III.}\ \eta^a_{ik} = \tfrac{1}{2}(\epsilon_{4ika} + \delta_{4i}\delta_{ka} - \delta_{4k}\delta_{ia}) \qquad \text{VI.}\ \eta^1_{ik} = R_{ik} \tag{8}$$

$$R_{ik} = \delta_{i1}\delta_{k2} - \delta_{k1}\delta_{i2}, \qquad S_{ik} = \delta_{i3}\delta_{k4} - \delta_{k3}\delta_{i4}.$$

The elements of the matrices η^a_{ik}, not indicated in formulas (8), are equal to zero. The five symmetrical zero-trace 3×3 matrices γ^{ab}_i satisfy the relations $Tr\,\gamma_i\gamma_k = \delta_{ik}$. The first three sets of matrices $i\eta^a_{ik}$ in formula (8) realized antisymmetrical representations of generators of the group $SU(2)$ with moments $T=2,1$ and $(\frac{1}{2}\oplus\frac{1}{2}^*)^{1)}$. In the case III, the representation is reducible and is given by t'Hooft matrices.[9]

The subsequent iterations of Eqs. (3) in each of the six cases give unambiguously the form of the solutions for arbitrary m/k. In cases II, III, V, and VI we have

$$A^a_i = \frac{1}{g}\eta^a_{ik} z_k a(S); \quad Y = \frac{u}{g}\phi(S); \quad S \equiv \eta^a_{ik}\eta^a_{ie} z_k z_e. \tag{9}$$

In case IV we obtain a solution that depends on two variables:

$$A^1_i = \frac{1}{g}\eta^1_{ik} z_k a_1(S_1, S_2); \qquad A^2_i = \frac{1}{g}\eta^2_{ik} z_k a_2(S_1, S_2); \qquad A^3_i = 0;$$
$$Y = \frac{u}{g}\phi(S_1, S_2); \qquad S_{1,2} \equiv \eta^{1,2}_{ik}\eta^{1,2}_{ie} z_k z_e. \tag{10}$$

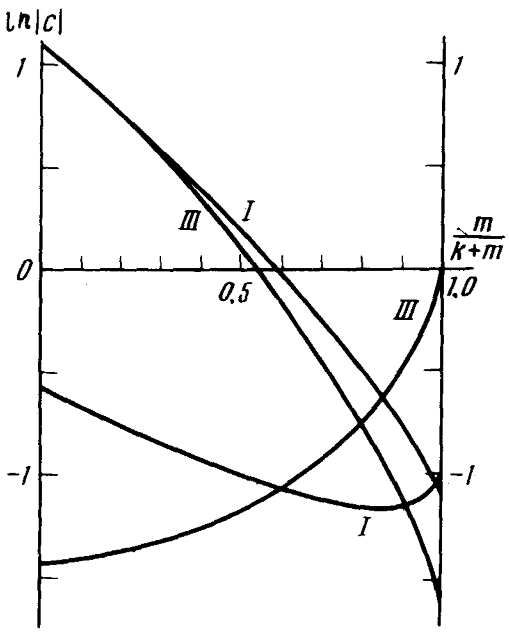

FIG. 1.

Finally, for the case I the solution takes the form

$$A_i^a = \frac{1}{g}\epsilon_{abc}\,[a_1(S)\hat{z}\gamma_i + a_2(S)\hat{z}^2\gamma_i + a_3(S)\hat{z}^2\gamma_i\hat{z}]_{fr}\,;$$
$$Y = \frac{u}{g}\,\phi(S);\quad \hat{z} \equiv z_i\gamma_i\,;\quad S \equiv \frac{1}{3}\mathrm{Sp}\,\hat{z}^3. \tag{11}$$

For the functions $a_i(S)$ and $\phi(S)$ we can in all cases write down a system of equations in closed form.

For case III it is possible to obtain an exact solution that goes over into (6a) and (6b) in the limit $m/k \ll 1$[1]. We write down this solution in four-dimensional form

$$A_\mu^a = -\frac{4}{g}\eta^a_{\mu\nu}x_\nu\frac{\rho^4 - 1}{(\rho^2 + x^2)(1 + \rho^2x^2)}\,;\qquad \phi = \frac{4i\sqrt{3}}{g}$$
$$\times\; u[(\rho^2 + x^2)(1 + \rho^2x^2)]^{-1/2}\,;\qquad \rho^4 = \frac{12\lambda}{g^2} - 1\,. \tag{12}$$

As seen from formula (12), A_μ^a coincides, apart from a factor, with a function that is a product of the solutions corresponding to instantons[9] with scales ρ and $1/\rho$.

The saddle-point values $\tilde{\lambda}$ and $\tilde{g}$ are determined from the condition that a functional (3) remains stationary when varied with respect to λ and g. It turns out here that at each fixed value of m and k there are two different sets of $\tilde{\lambda}$ and $\tilde{g}$. The coefficients of the expansion (2) are given at large k and m, accurate to the pre-exponential factor, by the formula[1]

$$G_n^{(k,m)}\cdot\;\mathrm{Re}\,e^{-J(\tilde{A},\tilde{\phi},\tilde{g},\tilde{\lambda})} \cong \mathrm{Re}\left(\frac{k+m}{16\pi^2e}\right)^{k+m}\left[C\left(\frac{m}{k+m}\right)\right]^{k+m}. \tag{13}$$

We must therefore select from among all the solutions the one for which ReJ is minimal, i.e., $|C|$ is maximal.

An iteration solution of the equations for the functions a_i and ϕ in formulas (9)–(11) enables us to find $|Cm/(k+m)|$ in the limit of small m/k,

$$\ln|C| = \ln 3 - \frac{m}{k+m}\ln 6 - B\left(\frac{m}{k+m}\right)^2 - \ldots, \tag{14}$$

where the coefficient B for cases I–VI is, respectively, B=(1/9, 10/63, 5/18, 5/42, 5/21, 5/28). Thus, the steepest descent in the functional integral at small m/k is given by solution I. The solutions of the equations for the cases I, II, and IV, just as the case III, contain ambiguities in $\tilde{\lambda}$ and $\tilde{g}$ expressed in terms of m and k.

As a rule, more convenient from the point of view of the steepness of the descent in the limit as $m/k \to \infty$ is the branch with $\tilde{\phi} \to 0$, i.e., in this limit we arrive at the case of the theory of a pure Yang-Mills field. We have obtained the extrema for all six forms of the solutions (9), (10), and (11) by using as trial functions the lower spherical harmonics on the five-dimensional sphere. Figure 1 shows the plots for the solutions I and III, which give the maximum values of $\ln|C|$ in formula (13). It is seen that in the limit as $m/k \to 0$ the most convenient solution is I, whereas at $m/k \to \infty$ the maximum contribution is made by the instanton-anti-instanton configuration used in[5]. The imaginary part of $\ln C$ determines the period of the oscillations of $G_n^{(k,m)}$. It turns out that at $m/k \ll 1$ the series (2) is of alternating sign both in g^2 and in λ, whereas at $m/k \gg 1$ the series (2) is of constant sign in g^2 and of alternating sign in λ. The pre-exponential factor in (13) is determined, as usual, by the number of tails n and by the number of zero modes that result from the breaking of the symmetry on the solutions (9)–(11).

We have thus obtaining in this paper all the possible forms of the solutions (9)–(11) of the classical equations for a model described by the Lagrangian (1), on the class of which is realized, in a sufficiently small interval of the values of the parameter m/k, the steepest descent in the functional integral for $G_n^{(k,m)}$. With increasing m/k, we cannot guarantee that the solution corresponding to the steepest descent is contained among the solutions of the form (9)–(11). It is interesting that the solution III, which is most convenient as $m/k \to \infty$, has the same character as the discontinuous solution used in[5] for a pure Yang-Mills field, but is contrariwise continuous.

The authors thank A. A. Belavin for useful discussions.

[1)]We note that as $\lambda/g^2 \to \infty$ the equations for case III have other solutions with practically the same action, differing from (12) mainly in the form of the solution for the scalar field ϕ.

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