New method in the theory of a weakly non-ideal one-dimensional Fermi gas

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A regular method is proposed for obtaining the energy of the ground state of the spectrum and the correlation functions of a weekly non-ideal one-dimensional Fermi gas.

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Despite the availability of many exact results in the problem of the non-ideal one-dimensional Fermi gas,[1-3] this problem is still far from its complete solution. In particular, if the interaction potential, even if small, is not δ-like, there is no regular method for calculating the energy, the spectrum, and the correlation functions as functions of the interaction constant. Methods of summing "parquet diagrams" and of the renormalization group make it possible to obtain only the leading terms in the corresponding quantities.[4,5] On the other hand, the reduction of the real spectrum to the one that is linear in the momentum[6-8] is not rigorous and (in any case) not angular for the calculation of the corresponding corrections. The purpose of the present communication is to develop a regular method of obtaining the corrections to the energy, spectrum, and correlation functions of a weakly non-ideal Fermi gas.

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We consider here a zero-spin Fermi gas and relegate the spin case to a more detailed paper.

We seek the wave function $\Psi(x_1,\dots,x_N)$ of a system of zero-spin Fermi particles with Hamiltonian

$$\hat{H} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i>j}^{N} v(x_i - x_j); \quad (\hbar = 2m = 1) \tag{1}$$

in the form

$$\Psi(x_1,\dots x_N) = \Psi_0(x_1,\dots x_N)\,\phi(x_1,\dots x_N), \tag{2}$$

where $\phi(x_1,\dots,x_N)$ is a symmetrical function and $\Psi_0(x_1,\dots,x_N)$ is the wave function of the ground state (1) at $V(x)=0$, apart from the normalization, is equal to (N is odd)

$$\Psi_0(x_1,\dots x_N) = \prod_{i>k} \sin\frac{\pi}{L}(x_i - x_k) \quad (L \text{ is the length of the system}). \tag{3}$$

Substituting (2) in the equation $\hat{H}\Psi=E\Psi$, we arrive at the equation

$$\tilde{H}\phi = (E - E_0)\,\phi, \tag{4}$$

where E_0 is the energy of the ground state (1) at $V(x)=0$, and $\tilde{H}$ is given by

$$\tilde{H} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - 2\frac{\pi}{L}\sum_{i>j} \operatorname{ctg}\frac{\pi}{L}(x_i - x_j)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right) + \sum_{i>j} V(x_i - x_j). \tag{5}$$

Since $\phi(x_1,\dots,x_N)$ is a symmetrical function, the problem of finding the wave function of the system of Fermi particles reduces in accordance with (4) to the corresponding problem for a system of bosons with Hamiltonian (5).

To solve (4) we use the regular method, proposed by Bogolyubov and Zubarev[9] and known from the theory of weakly nonideal Bose gas, of constructing the wave functions of the ground state and of weakly excited states. The applicability of this method to the one-dimensional problem was investigated by Popov.[10] In accordance with[9], in the approximation leading in $V(x)$, the function $\phi(x_1,\dots,x_N)$ for the ground state is given by

$$\phi(x_1,\dots x_N) = \exp\left\{\sum_{ij}^{N} S(x_i - x_j)\right\} \tag{6}$$

with

$$\sigma(k) = \pi(k^2 + 2p_F|k| - \sqrt{(k^2+2p_F|k|)^2 + 2\pi^{-1}p_F k^2 \nu(k)})/4p_F k^2;$$

$$\sigma(0) = 0, \tag{7}$$

where $\sigma(k)$ and $\nu(k)$ are the respective Fourier transforms of $S(x)$ and $V(x)$, and p_F is the Fermi momentum of the initial fermion system. For the ground-state energy E_{gr} for the spectrum $\epsilon(k)$ of the elementary excitations, which have a Bose character, we obtain

$$E_{gr} - E_0 = (2\pi)^{-1} p_F [N\nu(0) - \sum_{k=0} \nu(k) - 4\sum_k k^2 \sigma(k)],$$

$$\epsilon(k) = [(k^2 + 2p_F|k|)^2 + 2p_F \pi^{-1} k^2 \nu(k)]^{1/2}. \tag{8}$$

If we neglect the term k^2 in the combination $k^2+2p_F|k|$, then the equations in (8) coincide with the corresponding expressions obtained by Lieb and Mattis.[7] We note, however, that the results obtained by us are valid only at $\nu(k)/p_F \ll 1$. In particular, $\sigma(k)$ takes the form

$$\sigma(k) = -\nu(k)/4(k^2 + 2p_F|k|). \tag{9}$$

We can consider now the question of calculating the correlation function $g(x,x')=\langle\Psi|a^+(x)a(x')|\Psi\rangle$ in the ground state of (1) [$a^+(x)$ and $a(x)$ are the Fermi-particle creation and annihilation operators]. It can be shown that it takes the form

$$g(x,x') = \langle e^{T_0} a^+(x)a(x')e^{T_0}\rangle_0 / \langle e^{2T_0}\rangle_0, \tag{10a}$$

where $\langle\cdots\rangle_0$ denotes averaging with the wave function (3), and it is convenient to transform

$$\hat{T}_0 = \int_0^L dz \int_0^L dz' S(z-z')n(z)n(z'); \qquad n(z) = a^+(z)a(z) \tag{10b}$$

into

$$g(x,x') = \exp\{2S(0) - 2S(\zeta)\} \langle e^{2(T_0+T_1)} a^+(x)a(x')\rangle_0 / \langle e^{2T_0}\rangle_0, \tag{11}$$

where

$$\hat{T}_1 = \int_0^L [S(z-x') - S(z-x)]n(z)dz; \quad \zeta = x' - x,$$

and going from (10) to (11) we have used the fact that T_0 and T_1 commute. The calculation of (11) will be carried out by a diagram method. Expressing the mean value in (11) in terms of the contributions of only connected diagrams, we arrive at the relation (we use henceforth the momentum representation):

$$\langle e^{2(T_0+T_1)} a^+(x)a(x')\rangle_{0\,\mathrm{conn}} \Big/ e^{\sum_{n=1} \frac{2^n}{n!}\langle (T_0+T_1)^n\rangle_{0\,\mathrm{conn}}}$$

$$= g_0(\zeta) + \frac{1}{L}\sum_{p,q,n=1} e^{-ip\zeta+iqx} \frac{2^n}{n!}\langle (T_0+T_1)^n a^+_{p+q} a_p\rangle_{0\,\mathrm{conn}} \tag{12}$$

$g_0(\zeta)=(\pi\zeta)^{-1}\sin p_F\zeta$ is the correlation function of the ground state of (1) at $V(x)=0$. The diagrams corresponding to (12) consists of closed loops joined by the T_0 interaction lines. In each loop, summation is carried out over all the particle and hole lines and, in accordance with (12), two types of loop are possible, as shown graphically in Fig. 1. The contribution of a loop to which n lines of the interaction T_0 are connected is

$$\sum_{k=0} \frac{1}{k!} Q_{n+k,\alpha}(q_1,\dots q_n, p_1,\dots p_k)\tilde{\sigma}(p_1)\dots\tilde{\sigma}(p_k), \tag{13}$$

where $\alpha=1$ or 2 corresponds to Fig. 1(a) or 1(b), $\tilde{\sigma}(q)=\sigma(q)(\exp(iq\zeta)-1)$

$$Q_{m,1}(q_1,\dots q_m) = \sum_{p_1,\dots p_m} \langle a^+_{p_1+q_1} a_{p_1},\dots a^+_{p_m+q_m} a_{p_m} \rangle_{0\,\mathrm{conn}} \delta_{q_1+\dots+q_m,0},$$

$$Q_{m,2}(q_1,\dots q_m) = \sum_{p,q,p_1\dots p_m} e^{-ip\zeta} \langle a^+_{p_1+q_1} a_{p_1}\dots a^+_{p_m+q_m} a_{p_m} a^+_{p+q} a_p \rangle$$

$$_{0\,\mathrm{conn}}\delta_{q_1+\dots+q_m,-q}. \tag{14}$$

The terms $k\neq 0$ in (13) correspond to "intrusions" of an arbitrary number of operators T_1 into the loop. We present without proof the following statements concerning the properties of $g(x,x')=g(\zeta)$ and of the functions $Q_{n,\alpha}(q_1,\dots,q_n)$.

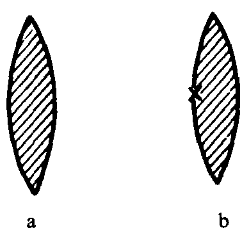

FIG. 1. Graphic representation of the loops.

I. In the calculation of $g(\zeta)$ as $\zeta\to\infty$ the only significant diagram contributions are those at $q_1,\dots,q_n\to 0$, i.e., the behavior of $g(\zeta)$ as $\zeta\to\infty$ is determined by the form of the functions $Q_{n,\alpha}(q_1,\dots,q_n)$ at small values of q (more accurately, at $|q_1|,\dots,|q_n|<2p_F$:

$$|q_1+q_2|, \dots |q_{n-1}+q_n| < 2p_F; \ |q_1+q_2+q_3| < 2p_F; \dots |q_1+\dots+q_n| < 2p_F).$$

This fact is connected with the divergence of $\sigma(q)$ as $q\to 0$.

II. At small q (in the sense indicated in Sec. I) we have

$$Q_{n,1}(q_1,\dots q_n) = \begin{cases} |q_1|, & n=2 \\ 0, & n>2 \end{cases};$$

$$Q_{n,2}(q_1,\dots q_n) = (i\zeta)^{-1}\prod_{i=1}^{n}(1-e^{-iq_i\zeta})[e^{ip_F\zeta}\prod_{i=1}^{n}\theta(q_i)$$

$$-e^{-ip_F\zeta}\prod_{i=1}^{n}\theta(-q_i)]; \quad \theta(x) = \begin{cases} 1, & x>0 \\ 0, & x<0 \end{cases}.$$

When II is taken into account, the series (12) can be summed exactly: the only diagrams that make a nonzero contribution are diagrams of the type shown in Fig. 2. Summing these diagrams we obtain for $g(\zeta)$:

$$g(\zeta) = g_0(\zeta)\exp\{-f_0(\zeta)\}; \; f_0(\zeta) = 2\pi^{-2}\int_0^\infty dq(1-\cos q\zeta)q\sigma^2(q)/(1-2q\pi^{-1}\sigma(q)). \tag{15}$$

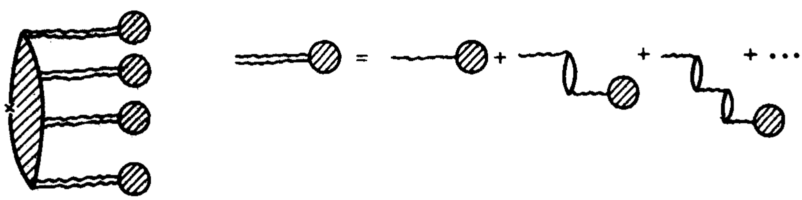

FIG. 2. Diagrams that make nonzero contributions to (12).

Inasmuch as the argument of the exponential in (15) is $\sim \ln p_F\zeta$ at $p_F\zeta \gg 1$, the distribution function $n_p = \langle\Psi|a_p^+ a_p|\Psi\rangle$ takes at $p \approx p_F$ the form

$$n_p = \frac{1}{2}\{1+\text{sign}(p_F-p)|p-p_F|^\beta\}; \qquad \beta = \nu^2(0)/32\pi^2 p_F^2. \tag{16}$$

A rigorous analysis of (12) shows that $g(\zeta)$ takes at $p_F\zeta \gg 1$ the form

$$g(\zeta) = \exp\{-f_0(\zeta)+f_1(\nu,\zeta)\}\{g_0(\zeta)+\zeta^{-1}\sum_{n=2}(\nu/p_F)^n\phi_n(\zeta)\}, \tag{17}$$

where $f_1(\nu,\zeta)$ and $\phi_n(\zeta)$ are the functions that are not singular as $\zeta\to\infty$. Equation (17) does not change the character of n_p in (16).

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